Online Appendix
The Stable Transformation Path

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There are seven sections in this appendix. Appendix A contains the proof of Theorem 1 in the main text. The derivation of the Euler Equation (14) in the main text is provided in Appendix B. We derive the dynamics of the system around the BGP in Appendix C. Appendix D provides a constructive proof of Proposition 3. Appendix E discusses our proposed algorithm for computing the STraP. Appendix F provides details of data construction for the empirical aspects of the paper in Section 5.

A Proof of Existence and Uniqueness of the STraP

This section provides a proof of Theorem 1 in the main text. The proof consists of two steps. The first step is standard, and we provide a succinct description (referring the reader to Acemoglu (2009) for details). The second step is novel, and we devote the majority of the proof to provide a detailed discussion.

The first step in proving our result is to characterize the solution of the Planner's problem going forward given capital k at time τ. To apply the standard results from optimal control, we need to de-trend our variables so that they are bounded. To do that, we will define two effective productivity terms. We define $\mathcal{A}_k(t)$ implicitly by

$$1 = \sum_j \omega_{xj} \left(\frac{\mathcal{A}_k(t)^{1-\alpha_j}}{A_x(t) A_j(t)} \right)^{1-\sigma_x}.$$

This productivity term captures how capital and investment grows. We define an

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implicit productivity term for consumption $\mathcal{A}_c(t)$ similarly

$$1 = \sum_j \omega_{cj} \left(\frac{\mathcal{A}_c(t)^{\varepsilon_j}}{\mathcal{A}_k(t)^{\alpha_j} A_j(t)} \right)^{1-\sigma_c}.$$

We can then rewrite the consumer problem in terms of the bounded variables $k(t) \equiv \frac{K(t)}{\mathcal{A}_k(t)}$, $x(t) \equiv \frac{X(t)}{\mathcal{A}_k(t)}$ and $c(t) \equiv \frac{C(t)}{\mathcal{A}_c(t)}$. For our purposes, it will be easier to work with the planner's problem. Define $c(x,k,t)$ as the maximum de-trended consumption that the planner could get at time t with detrended capital k and detrended investment x. This corresponds with the competitive equilibrium and is implicitly defined, along with e, p_j, r, w, by the equations

$$rk = \sum_j \alpha_j \left[\omega_{xj} \left(\frac{\mathcal{A}_k(t)^{1-\alpha_j}}{A_x(t) A_j(t)} \right)^{1-\sigma_x} p_j^{1-\sigma_x} x + \omega_{cj} \left(\frac{\mathcal{A}_c(t)^{\varepsilon_j}}{\mathcal{A}_k(t)^{\alpha_j} A_j(t)} \right)^{1-\sigma_c} p_j^{1-\sigma_c} c^{\varepsilon_j(1-\sigma_c)} e^{\sigma_c} \right]$$

$$wL = \sum_j (1-\alpha_j) \left[\omega_{xj} \left(\frac{\mathcal{A}_k(t)^{1-\alpha_j}}{A_x(t) A_j(t)} \right)^{1-\sigma_x} p_j^{1-\sigma_x} x + \omega_{cj} \left(\frac{\mathcal{A}_c(t)^{\varepsilon_j}}{\mathcal{A}_k(t)^{\alpha_j} A_j(t)} \right)^{1-\sigma_c} p_j^{1-\sigma_c} c^{\varepsilon_j(1-\sigma_c)} e^{\sigma_c} \right]$$

$$p_j = \left(\frac{r}{\alpha_j} \right)^{\alpha_j} \left(\frac{w}{1-\alpha_j} \right)^{1-\alpha_j}$$

$$1 = \sum_j \omega_{xj} \left(\frac{\mathcal{A}_k(t)^{1-\alpha_j}}{A_x(t) A_j(t)} \right)^{1-\sigma_x} p_j^{1-\sigma_x}.$$

$$e^{1-\sigma_c} = \sum_j \omega_{cj} \left(\frac{\mathcal{A}_c(t)^{\varepsilon_j}}{\mathcal{A}_k(t)^{\alpha_j} A_j(t)} \right)^{1-\sigma_c} p_j^{1-\sigma_c} c^{\varepsilon_j(1-\sigma_c)}.$$

Each of these equations are smooth i.e., infinitely continuously differentiable, so by the implicit function theorem, $c(x,k,t)$ is also smooth in a neighborhood around any point.

Then we can write the current value hamiltonian

$$\mathcal{A}_c(t)^{1-\theta} u(x(t), k(t), t) - \frac{1}{1-\theta} + \mu(t) \left[x(t) - (\delta + \gamma_k(t)) k(t) \right]$$

where $\gamma_k(t) \equiv \frac{\dot{\mathcal{A}}_k(t)}{\mathcal{A}_k(t)}$, and $u(x,k,t) \equiv \frac{c(x,k,t)^{1-\theta}}{1-\theta}$.

The assumptions on preferences and production then guarantee the solution is interior. Moreover, Theorem 7.9 in Acemoglu (2009) holds, so any solution must satisfy the first order conditions of the Hamiltonian:

$$0 = \mathcal{A}_c(t)^{1-\theta} \frac{\partial u(x,k,t)}{\partial x} + \mu(t)$$

$$\rho\mu(t) - \dot{\mu}(t) = \mathcal{A}_c(t)^{1-\theta} \frac{\partial u(x,k,t)}{\partial k} - \mu(t)(\delta + \gamma_k(t)).$$

We can then solve these to get an Euler equation for investment:

$$\frac{\dot{x}(t)}{x(t)} = \frac{1}{\theta(x,k,t)}\left[\frac{\frac{\partial u}{\partial k}}{\frac{\partial u}{\partial x}} + \delta + \rho + \gamma_k(t) + (1-\theta)\gamma_c(t) - \frac{\frac{\partial^2 u}{\partial x \partial k}}{\frac{\partial u}{\partial x}}\left(x - (\delta + \gamma_k(t))k\right) - \frac{\frac{\partial^2 u}{\partial x \partial t}}{\frac{\partial u}{\partial x}}\right], \tag{A.1}$$

where $\theta(x,k,t) = \frac{\frac{\partial^2 u}{\partial x^2}x}{\frac{\partial u}{\partial x}}$ and $\gamma_c(t) \equiv \frac{\dot{\mathcal{A}}_c(t)}{\mathcal{A}_c(t)}$. The system is further characterized by the law of motion

$$\dot{k}(t) = x(t) - (\delta + \gamma_k(t))\, k(t) \tag{A.2}$$

Theorem 7.12 in Acemoglu (2009) holds (since the value function is differentiable and its derivative converges to 0 along any feasible path), implying that any interior solution path satisfies the transversality condition

$$\lim_{t\to\infty} H(t, k(t), x(t), \lambda(t)) = 0. \tag{A.3}$$

Finally, the maximized Hamiltonian is strictly concave in the state variable so the continuous path defined by the law of motion (equation A.2), the Euler equation (equation A.1), and the transversality condition (equation A.3) is the unique solution to the planner's problem.

Now we proceed to the second step of the proof. In this part, we need to ensure that the system is Lipschitz continuous to invoke some of the results from the theory of ordinary differential equations. This requires that the production functions are Lipschitz continuous. We therefore leverage the fact that they satisfy the Inada conditions and define an $\underline{\varepsilon} > 0$ that bounds capital away from 0 with the property that if $k(\tau) \in [\underline{\varepsilon}, \bar{k}]$, then $k(t) \in [\underline{\varepsilon}, \bar{k}]$ for all $t > \tau$. Given that the production functions are C^2 for all $(k,t) \in (0,\infty) \times \mathbb{R}$, it is Lipschitz continuous in $[\underline{\varepsilon}, \bar{k}]$ and we focus on that interval. We also define an $\underline{X}(k)$ and $\overline{X}(k)$ so that for all $t \in \mathbb{R}$ and $k \in [\underline{\varepsilon}, \bar{k}]$, the optimal investment is always in $x(t) \in (\underline{X}(k), \overline{X}(k))$ and that implies that $c(x,k,t)$ is bounded away from 0. Then, we can restrict the domain of investment to the compact interval $[\underline{X}(k), \overline{X}(k)]$ without loss.

The Euler equation and the law of motion for capital determine a two dimensional non-autonomous system. In the first step of our proof, we have shown that there is a unique consumption and investment level for a given starting capital k and time t consistent with optimization. This investment function, which we label $x(k,t)$, is the unique investment level that shoots to the asymptotic balanced growth path with capital level k_∞. This allows us to write our system as a one dimensional non-autonomous differential equation

$$\dot{k}(t) = x(k(t), t) - (\delta + \gamma_k(t))\, k(t). \tag{A.4}$$

By construction, given initial conditions $(k_0, \tau) \in [\underline{\varepsilon}, \bar{k}] \times \mathbb{R}$, $k(t) \to k_\infty$ as $t \to \infty$. Proving existence and uniqueness of the STraP comes down to proving that there exists one unique path of this system that has $k(t) \to k_{-\infty}$ as $t \to -\infty$. To do that, we will use the anti-funnel existence and uniqueness Theorem 4.7.5 from Hubbard and West (1991).

Let us outline how the rest of the proof proceeds. At a broad level, the main complication in this step is characterizing $x(k,t)$. Define $c_{-}(k), x_{-}(k)$ as the optimal choice of consumption and investment respectively given a starting capital amount in the negative asymptotic optimal growth problem. Define $c_{+}(k), x_{+}(k)$ similarly for the positive asymptotic optimal growth problem. The usual growth phase plane analysis gives us information about $c_{-}(k)$ and $c_{+}(k)$. For example, linearizing $c_{-}(k)$ around $k_{-\infty}$ tells us that c is continuously differentiable and increasing in k around the steady state. In Lemma A.1, we prove that these properties extend to $c(k,t)$ and $x(k,t)$ for t close enough to $-\infty$. We use this result to check some of the conditions needed for the anti-funnel theorem in the ensuing lemmas. In Lemma A.3, we show that equation (A.4) gets close to the law of motion for the negative asymptotic optimal growth problem for $t \to -\infty$. Then using these results, we construct an anti-funnel and apply the anti-funnel theorem in Proposition A.1.

We begin by characterizing the asymptotic behavior of $x(k,t)$. We reparametrize time to lie in the compact interval $[0,1]$ where 0 corresponds to $t=-\infty$ and 1 corresponds to $t=\infty$. Then we can use continuity to extend results about functions at $t=\pm\infty$ to functions where $t \in \mathbb{R}$. We will define the function $h : \mathbb{R} \to (0,1)$ as $h(t) = g(e^t/(1+e^t))$ where

$$g(x) = (1-x)x^{\frac{1}{2}} + x\left(1-(1-x)^{\frac{1}{2}}\right).$$

$e^t/(1+e^t)$ takes $t \in (-\infty,\infty)$ and maps it to $(0,1)$. $g(\cdot)$ is an monotonically increasing function that looks like $x^{\frac{1}{2}}$ as $x \to 0$ and $1-(1-x)^{\frac{1}{2}}$ as $x \to 1$. h will then be our way of mapping time to the interval $[0,1]$ and the g adjustment simplifies some of the limits.

Lemma A.1. *There exists a function* $\check{x} : \left[\underline{\varepsilon}, \bar{k}\right] \times [0,1] \to [-\underline{X}(k), \overline{X}(k)]$ *with the following properties:*

- $\check{x}(k,0) = x_{-}(k)$ *for all* $k \in [\underline{\varepsilon}, \bar{k}]$,
- $\check{x}(k,1) = x_{+}(k)$ *for all* $k \in [\underline{\varepsilon}, \bar{k}]$,
- $\check{x}(k,h(t)) = x(k,t)$ *for all* $k \in [\underline{\varepsilon}, \bar{k}]$ *and* $t \in (-\infty,\infty)$, *and*
- $\check{x}(k,z)$ *is continuously differentiable.*

Proof. It is easy to construct a function $\check{x}(k,z)$ that satisfies the first three properties simply by fiat. The only thing to note is that this must be a function since our maximization theorem guarantee a unique maximum.

The harder step is proving that $\check{x}(k,z)$ is continuously differentiable. We will do that by showing that it is a manifold defined by a continuously differentiable ordinary differential equation system in a compact set.

Before we do that, we will start by proving that the system is continuous. In order to do this, we will apply Berge's maximum theorem. We can rewrite the planner's maximization problem as

$$\mathcal{W}(z) = \max_{x(t),k(t)} \int_0^\infty e^{-\rho(t+h^{-1}(z))} \left(\frac{\mathcal{A}_c(t+h^{-1}(z))}{\mathcal{A}_c(h^{-1}(z))} \right)^{1-\theta} u(x(t), k(t), t + h^{-1}(z))dt$$

such that

$$\dot{k}(t) = x(t) - (\delta + \gamma_k(t + h^{-1}(z)))k(t)$$

and

$$k(0) = k_0$$

where we have simply done a change of variables on starting time τ to z and divided by $\mathcal{A}_c(\tau)$. Berge's maximum theorem then guarantees that the choice variables $x(t)$ and $k(t)$ are continuous in z if the conditions are satisfied. That would then guarantee that $\check{x}(k, z)$ is continuous, even as $z \to 0, 1$.

The conditions are that the utility function are continuous and the constraint correspondence is continuous, compact valued, and contains no empty values. We will use the discounted norm $||z(\cdot)|| \equiv \int_0^\infty e^{-\rho t} z(t)dt$. $\mathcal{A}_c(t)$ grows at the rate $\frac{1}{1-\alpha_a} \frac{1-\theta}{\varepsilon_a}[\alpha_a\gamma_x + \gamma_a]$ in the $-\infty$ limit and the rate $\frac{1}{1-\alpha_s} \frac{1-\theta}{\varepsilon_s}[\alpha_s\gamma_x + \gamma_s]$ in the $+\infty$ limit. Then as long as ρ is greater than both of those, we can use the dominated convergence theorem and the continuity of the integrand implies the continuity of the integral. Therefore, Berge's theorem holds and $\check{x}(k, z)$ is continuous in its arguments.

Next, we need to prove that $\check{x}(k, z)$ is continuously differentiable. We do this by considering the differential equations that define the solution to the planner's problem. We will show that $\check{x}(k, z)$ is a stable manifold in this system, and so, will be continuously differentiable if the system is nice. We can write them as a 3-dimensional autonomous system by introducing a variable s that stands in for time:

$$\dot{k}(t) = x - (\delta + \gamma_k(s))\, k, \tag{A.5}$$

$$\dot{x}(t) = \frac{x}{\theta(x,k,s)} \Big[- R(x,k,s) + \delta + \rho + \gamma_k(s) + (1-\theta)\gamma_c(s) + \zeta_k(x,k,s)\,(x - (\delta + \gamma_k(s))k) + \zeta_t(x,k,s) \Big], \tag{A.6}$$

$$\dot{s}(t) = 1 \tag{A.7}$$

where $R(x,k,t) = -\frac{\frac{\partial u}{\partial k}}{\frac{\partial u}{\partial x}}$, $\zeta_k(x,k,t) = -\frac{\frac{\partial^2 u}{\partial x \partial k}}{\frac{\partial u}{\partial x}}$, and $\zeta_t(z,k,t) = -\frac{\frac{\partial^2 u}{\partial x \partial t}}{\frac{\partial u}{\partial x}}$ The variable s is unbounded in this system which makes it cumbersome to deal with. Therefore, we reparametrize time with $z = h(s) \in (0,1)$. This autonomous system can be written

$$\dot{k} = x - \big(\delta + \gamma_k(h^{-1}(z))\big)\, k, \tag{A.8}$$

$$\begin{aligned}\dot{x} = \frac{x}{\theta(x,k,h^{-1}(z))}\big[&- R(x,k,h^{-1}(z)) + \delta + \rho + \gamma_k(h^{-1}(z)) + (1-\theta)\gamma_c(h^{-1}(z)) \\ &+ \zeta_k(x,k,h^{-1}(z))(x - \delta k - \gamma_k(h^{-1}(z))k) + \zeta_t(x,k,h^{-1}(z))\big],\end{aligned} \tag{A.9}$$

$$\dot{z} = h'(h^{-1}(z)), \tag{A.10}$$

and it is defined on the set $[\varepsilon,\bar{k}]\times[\underline{X}(k),\overline{X}(k)]\times(0,1)$. We denote the right hand side of equations (A.8), (A.9) and (A.10) by $F_k(k,x,z)$, $F_x(k,x,z)$, and $F_z(k,x,z)$, respectively. The system can easily be extended to the compact set $[\varepsilon,\bar{k}]\times[\underline{X}(k),\overline{X}(k)]\times[0,1]$ by replacing $\gamma_k(t)$, $\gamma_c(t)$, $R(x,k,t)$, $\zeta_k(x,k,t)$, $\zeta_t(x,k,t)$, $\theta(x,k,t)$ and $h'(t)$ with their limits as $z \to 0, 1$. To do that, we need to confirm that those limits exist.

Proving the existence of the limits $\gamma_k(t)$ and $\gamma_c(t)$ follows pretty quickly from implicitly differentiating the expressions for $\mathcal{A}_k(t)$ and $\mathcal{A}_c(t)$. The limits on the other terms is more difficult because they are multiple derivatives of the complicated function $u(x,k,t)$. Instead what we do is notice that it is very easy to rewrite the function $u(x,k,t)$ as a function of z, $u(x,k,z)$. Then since the function is implicitly defined by smooth functions on the whole interval $z \in [0,1]$, $u(x,k,z)$ is also a smooth function defined on all (x,k,z). Therefore, the limits of $R(x,k,z)$, $\zeta_k(x,k,z)$, $\zeta_t(x,k,z)$, $\theta(x,k,z)$ all exist correspond with their values, and the derivatives converge as well. Since $h(t)$ converges to a real number and h' converges, it follows that h' must converge to 0 as $t \to \pm\infty$.

Next, we can take the total differential of the function F_k, F_x, and F_z to get

$$dF_k(k,x,z) = -\left[\delta + \gamma_k(h^{-1}(z))\right] dk + dx - \dot{\gamma}_k(h^{-1}(z))k \left(h^{-1}\right)'(z)dz, \tag{A.11}$$

$$\begin{aligned} dF_x(k,x,z) = {} & \frac{x}{\theta(x,k,h^{-1}(z))}\Bigg[-\frac{\partial R}{\partial k} + \frac{\partial \zeta_k}{\partial k}(x - \delta k - \gamma_k(h^{-1}(z))k) \\ & - \zeta_k(\delta + \gamma_k(h^{-1}(z))) + \frac{\partial \zeta_t}{\partial k} - F_x(k,x,z)\frac{\partial \theta}{\partial k}\Bigg] dk \\ & + \frac{x}{\theta(x,k,h^{-1}(z))}\Bigg[-\frac{\partial R}{\partial x} + \frac{\partial \zeta_k}{\partial x}(x - \delta k - \gamma_k(h^{-1}(z))k) + \zeta_k \\ & + \frac{\partial \zeta_t}{\partial x} - F_x(k,x,z)\frac{\partial \theta}{\partial x} + \frac{1}{x}F_x(k,x,z)\Bigg] dx \\ & + \frac{x}{\theta(x,k,h^{-1}(z))}\Bigg[-\frac{\partial R}{\partial t} + \dot{\gamma}_k + \frac{\partial \zeta_k}{\partial t}(x - \delta k - \gamma_k(h^{-1}(z)))k - \zeta_k \dot{\gamma}_k \\ & + \frac{\partial \zeta_t}{\partial t} - F_x(k,x,z)\frac{\partial \theta}{\partial t}\Bigg] (h^{-1})'(z)dz, \end{aligned} \tag{A.12}$$

$$dF_z(k,c,z) = h''(h^{-1}(z))(h^{-1})'(z)dz. \tag{A.13}$$

These derivatives all exist, they are continuous, and k, x, and z are all bounded in a compact set. Therefore, the original system must be lipschitz continuous.

The next step is finding our function $\check{x}(k,z)$ in this system. Notice that there are two steady states with positive consumption in the system defined by equations (A.8), (A.9) and (A.10): one at $(k_\infty, x_\infty, 1)$ and another at $(k_{-\infty}, x_{-\infty}, 0)$. Next, we linearize this system of equations around the $(k_\infty, x_\infty, 1)$ steady state. Then a deviation $k_\infty + \hat{k}(t)$, $x_\infty + \hat{x}(t)$, and $1 + \hat{z}(t)$ must locally satisfy

$$\begin{pmatrix} \dot{\hat{k}}(t) \\ \dot{\hat{x}}(t) \\ \dot{\hat{z}}(t) \end{pmatrix} = \begin{pmatrix} -\delta - \gamma_{k+} & 1 & 0 \\ \frac{c_\infty}{1-\frac{1-\theta}{\varepsilon_j}}\alpha_j(1-\alpha_j)\frac{y_\infty}{k_\infty^2} + \alpha_j \frac{y_\infty}{k_\infty^2} x_\infty & -\alpha_j \frac{y_\infty}{k_\infty} & 0 \\ 0 & 0 & a_+ \end{pmatrix} \begin{pmatrix} \hat{k}(t) \\ \hat{x}(t) \\ \hat{z}(t) \end{pmatrix}. \tag{A.14}$$

The associated characteristic polynomial to (A.14) is

$$(a_+ - \lambda)\left[(\delta + \gamma_{k+} + \lambda)(\alpha_j \frac{y_\infty}{k_\infty} + \lambda) - \frac{c_\infty}{1 - \frac{1-\theta}{\varepsilon_j}}\alpha_j(1-\alpha_j)\frac{y_\infty}{k_\infty^2} + \alpha_j \frac{y_\infty}{k_\infty^2} x_\infty\right].$$

The system has one positive and two negative eigenvalues. Thus, locally this stable plane defines investment as a function of k and z.

We can then extend this to a manifold in the set $[\varepsilon, \bar{k}] \times [\underline{X}(k), \overline{X}(k)] \times (0, 1]$. This manifold is the set of (x, k, z) that shoot to the asymptotic BGP. Clearly, our optimally chose $x(k,t)$ which is the same as $\check{x}(k,z)$ must be on this manifold. And by uniqueness, that is the only value of x on the manifold for a given k z. And furthermore, we know that taking the limit as $z \to 0$, $\check{x}(k,z)$ converges to $\check{x}(k,0)$. Therefore, the manifold also converges to that, and we can extend the manifold to $[\varepsilon, \bar{k}] \times [\underline{X}(k), \overline{X}(k)] \times [0, 1]$. Because the system is Lipschitz continuous, the manifold must be continuously differentiable. Therfore, $\check{x}(k,z)$ must be continuously differentiable since it is the manifold. □

We next present a series of Lemmas that build on Lemma A.1 that allow us to show that the conditions for Theorem 4.7.5 of Hubbard and West (1991) hold in this setting.[1] To use the anti-funnel theorem in Hubbard and West, we will look at the

[1] For completeness, we state Theorem 4.7.5 from Hubbard and West (1991) here.

Theorem A.1. *Let $\alpha(t)$ and $\beta(t)$, $\beta(t) \leq \alpha(t)$, be two fences defined for $t \in [a, b)$ that bound an antifunnel for the differential equation $x' = f(t, x)$. Let $f(t, x)$ satisfy a Lipschitz condition in the antifunnel. Furthermore, let the antifunnel be narrowing, with*

$$\lim_{t \to b} (\alpha(t) - \beta(t)) = 0.$$

If $(\partial f/\partial x)(t, x) \geq w(t)$ in the antifunnel, where $w(t)$ is a function satisfying

$$\int_a^b w(s)ds > -\infty,$$

then there is a unique solution which stays in the antifunnel.

backwards differential equation. Define $\tilde{k}(t) = k(-t)$. This follows the differential equation

$$\dot{\tilde{k}}(t) = -x(\tilde{k}(t), t) + (\delta + \gamma_k(-t))\,\tilde{k}(t). \tag{A.15}$$

We then want to show that there exists a unique time path $\tilde{k}(t)$ with $\tilde{k}(t) \to k_{-\infty}$ as $t \to \infty$. Define $F\left(\tilde{k}, t\right) \equiv -x(\tilde{k}(t), t) + (\delta + \gamma_k(-t))\,\tilde{k}(t)$. We similarly define the function that describes the backward capital motion in the negative asymptotic growth problem: $F_-\left(\tilde{k}\right) \equiv -x(\tilde{k}(t)) + (\delta + \gamma_{-k})\,\tilde{k}(t)$.

Lemma A.2. *On some interval $U \subset [0, \bar{k}]$ containing $k_{-\infty}$ in its interior, for all $\varepsilon > 0$, there exists a $T > 0$ so that if $t > T$,*

$$\left|F\left(\tilde{k}, -t\right) - F_-\left(\tilde{k}\right)\right| < \varepsilon, \qquad \forall \tilde{k} \in I.$$

Proof. By Lemma A.1, $\check{x}(k, z)$ is continuous on the compact region $[\varepsilon, \bar{k}] \times [0, 1]$. Therefore, it is absolutely continuous. Then $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that $|\check{x}(k_1, z_1) - \check{x}(k_2, z_2)| < \varepsilon$, if $\max\{|k_1 - k_2|, |z_1 - z_2|\} < \delta$, where we use the sup norm. In particular, if $z_2 = 0$ and $k_1 = k_2 \equiv k$, then for $z_1 < \delta$, $|\check{x}(k, z_1) - \check{x}(k, 0)| < \varepsilon$. Transforming into time, if $t_1 < h(\delta)$ then

$$|x(k, t_1) - x_-(k)| < \varepsilon, \forall k \in [\varepsilon, \bar{k}].$$

To finish the proof, we simply note that

$$\begin{aligned}
\left|F\left(\tilde{k}, -t\right) - F_-\left(\tilde{k}\right)\right| &= \left|-x\left(\tilde{k}, -t\right) + x_-\left(\tilde{k}\right) + +\gamma_k(-t)\tilde{k} - \gamma_{-k}\tilde{k}\right| \\
&\leq \varepsilon + |\gamma_k(-t) - \gamma_{-k}|\,\tilde{k} \\
&\leq 2\varepsilon
\end{aligned}$$

for t sufficiently large that $|\gamma_k(-t) - \gamma_{-k}| < \frac{\varepsilon}{\bar{k}}$. □

Lemma A.3. $F\left(\tilde{k}, t\right)$ *is Lipschitz continuous on some interval U containing $k_{-\infty}$ in its interior. That is, there exists a $\mathcal{K} > 0$ such that for all $t \in [0, \infty)$ and $\tilde{k}_1, \tilde{k}_2 \in I$,*

$$\left|F\left(\tilde{k}_1, t\right) - F\left(\tilde{k}_2, t\right)\right| \leq \mathcal{K}\left|\tilde{k}_1 - \tilde{k}_2\right|.$$

Proof. Notice that $\check{x}(k, z)$ is continuously differentiable. Therefore, $\left|\frac{\partial \check{x}(k,z)}{\partial k}\right|$ is continuous on the compact interval $[\varepsilon, \bar{k}] \times [0, 1]$ and is bounded above by some M_1. Define $\bar{\gamma}_k \equiv \sup_{t\in\mathbb{R}} \gamma_k(t)$. Then for any t and $\tilde{k}_1, \tilde{k}_2 \in I$,

$$\begin{aligned}
\left|F\left(\tilde{k}_1, t\right) - F\left(\tilde{k}_2, t\right)\right| &\leq |\delta + \gamma_k(-t)| \cdot \left|\tilde{k}_1 - \tilde{k}_2\right| + \left|\check{x}\left(\tilde{k}_1, h(-t)\right) - \check{x}\left(\tilde{k}_2, h(-t)\right)\right| \\
&\leq 2 \cdot \max\{M_1, |\delta + \bar{\gamma}_k|\}\left|\tilde{k}_1 - \tilde{k}_2\right|
\end{aligned}$$

Taking $\mathcal{K} \equiv d \cdot \max\{M_1, |\delta + \bar{\gamma}_k|\}$ completes the proof. □

Lemma A.4. *There exists an interval U containing $k_{-\infty}$ in its interior, and a function $w : [a, \infty) \to \mathbb{R}$ such that*

$$\frac{\partial F(k,t)}{\partial k} \geq w(t)$$

for $k \in U$ where $\int_a^\infty w(s)ds > -\infty$.

Proof. Notice that

$$\begin{aligned}\frac{\partial F\left(k,t\right)}{\partial k} &= \frac{\partial}{\partial k}\left[-x\left(\tilde{k},-t\right)+\left(\delta+\gamma_k\left(-t\right)\right)\tilde{k}\right] \\ &= -\frac{\partial x(\tilde{k},-t)}{\partial k}+\delta+\gamma_k(-t)\end{aligned}$$

If one linearizes $x_-(k)$ around $k_{-\infty}$, one can see from the stable arm that

$$\frac{\partial F(k_{-\infty},\infty)}{\partial k} > 0$$

Then by continuity, there exists a $T > 0$ and $\delta > 0$ so that for $k \in (k_{-\infty} - \delta, k_{-\infty} + \delta)$ and $t > T$, $\frac{\partial F(k,t)}{\partial k} > 0$. Therefore, we can take $U = (k_{-\infty} - \delta, k_{-\infty} + \delta)$ and $w(t) = \min\left\{\inf_{k \in U} \frac{\partial F(k,t)}{\partial k}, 0\right\}$. Then $w(t) = 0$ for all $t > T$ so that $\int_a^\infty w(s)ds > -\infty$. □

Proposition A.1. *There exists a unique STraP.*

Proof. We use Theorem 4.7.5 from Hubbard and West (1991) stated in Footnote 1 to show existence and uniqueness of the STraP. Lemma A.3 showed that the system is Lipschitz. We need to construct the narrowing upper and lower fence. We restrict attention to a symmetric interval around $k_{-\infty}$ where Lemmas A.2 through A.4 hold. Define this as $U \equiv [k_{-\infty} - \delta, k_{-\infty} + \delta]$. This is possible as $k_{-\infty}$ was in the interior of all of the intervals.

We describe the explicit construction of the upper fence. The construction of the lower fence proceeds analogously by symmetry. Define $k_0 \equiv k_{-\infty} + \delta$, $k_1 \equiv k_{-\infty} + \frac{\delta}{2}, \cdots,\ k_n \equiv k_{-\infty} + \frac{\delta}{2^n}$ where $n \in \mathbb{Z}_0$. Take $a_n \equiv F_-(k_n)$. Standard growth dynamics tell us that $F_-(k)$ is increasing in k (recall that $F_-(k)$ is the backwards dynamics). Therefore, monotonicity implies that $F_-(k) > a_n$ if $k > k_n$. This also implies that $a_n > 0$ for all n since $F_-(k_{-\infty}) = 0$. By Lemma A.3, $F\left(\tilde{k}, t\right)$ gets arbitrarily close to $F_-(k)$. Therefore, for every n, there exists a $T_n > 0$ so that for $t > T_n$,

$$\left|F\left(\tilde{k},t\right) - F_-\left(\tilde{k}\right)\right| < \frac{a_n}{2}$$

for all $\tilde{k} \in U$.

We can now describe the explicit construction of the upper fence, $\alpha(t)$. Let $\alpha(t)$ be a piecewise linear function starting at time T_1 with $\alpha(T_1) = k_0$ and proceeding linearly to $\alpha(T_2) = k_1$. Then the derivative of α is negative. Furthermore,

$$|F(\alpha(t), t) - F_{-}(\alpha(t))| < \frac{a_1}{2}$$

and

$$F_{-}(\alpha(t)) > F_{-}(k_1) = a_1,$$

for all $t \in [T_1, T_2]$. Therefore, $F(\alpha(t), t) > 0$ for all $t \in [T_1, T_2]$ and $\alpha(\cdot)$ counts as a fence on this interval. Continue concatenating linear functions like this with $\alpha(T_n) = k_{n-1}$ for all n. $\alpha'(t) < 0$ for all t and $F(\alpha(t), t) > 0$ for all t. Therefore, this piecewise function is a fence converging to $k_{-\infty}$.

Analogously, we can construct a lower fence $\beta(t)$ which converges to $k_{-\infty}$. We have constructed $\alpha(t)$ and $\beta(t)$ that bound an anti-funnel. The final condition is provided by Lemma A.4. Thus, we can apply the theorem obtaining the desired result: there is a unique time path that remains in the anti-funnel. This is the STraP. □

B Derivation of the Euler Equation (14)

Omitting explicit time dependence, we can write the current-value Hamiltonian as

$$\max_{C,X,K} \frac{C^{\theta}}{1-\theta} + \lambda \left(WL + RK - \left(\sum_{i \in a,m,s} \omega_{ci} \left(P_i C^{\epsilon_i} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}} - P_x X \right) + \mu \left(X - \delta K \right) \tag{B.1}$$

where we have used the solution for the expenditure function E of nonhomothetic CES (Comin et al., 2021) in the budget constraint. Denoting by $\mathcal{H}$ the Hamiltonian, the first-order conditions imply:

$$\frac{d\mathcal{H}}{dC} = 0 \Longrightarrow C^{-\theta} = \lambda \frac{dE}{dC} = \lambda \bar{\epsilon} P_c \tag{B.2}$$

$$\rightarrow -\theta \frac{\dot{C}}{C} = \frac{\dot{\lambda}}{\lambda} + \frac{\dot{\bar{\epsilon}}}{\bar{\epsilon}} + \frac{\dot{P}_c}{P_c} \tag{B.3}$$

$$\frac{d\mathcal{H}}{dX} = 0 \Longrightarrow P_x \lambda = \mu \tag{B.4}$$

$$\rightarrow \frac{\dot{P}_x}{P_x} + \frac{\dot{\lambda}}{\lambda} = \frac{\dot{\mu}}{\mu} \tag{B.5}$$

$$\frac{d\mathcal{H}}{dK} = -\dot{\mu} + \rho\mu \Longrightarrow \lambda R + \dot{\mu} = (\rho + \delta)\mu \tag{B.6}$$

$$\rightarrow \frac{\dot{\mu}}{\mu} = \rho + \delta - \frac{R}{P_x} \tag{B.7}$$

We have used in (B.2) that $\frac{d\ln E}{d\ln C} = \sum_i \frac{P_i C_i}{E}\epsilon_i$ and the definition $P_c \equiv E/C$. Substituting (B.7) and (B.5) in (B.4) yields the Euler Equation. To obtain Equation (14) we simply need to rewrite nominal expenditures to obtain C in units of the investment good.

C Local Dynamics around the Asymptotic BGP

We characterize the dynamics of normalized variables, where the normalizing factor is $\mathcal{A}_x(t)^{1/(1-\alpha)}$, and we use lowercase to indicate normalized variables. This normalization and notational convention will continue throughout the paper. In terms of the normalized variables, the local dynamics of the economy in the neighborhood of the asymptotic BGP are given by the following system of ordinary differential equations:

$$\theta\frac{\dot{c}(t)}{c(t)} = \alpha k(t)^{\alpha-1} - \delta - \rho + (1-\theta)\left(\bar{\gamma}_c(t) - \gamma_x - \bar{\gamma}_x(t)\right) - \theta\frac{\gamma_x + \bar{\gamma}_x(t)}{1-\alpha}, \tag{C.1}$$

$$\dot{k}(t) = k(t)^{\alpha} - c(t) - \left(\delta + \frac{\gamma_x + \bar{\gamma}_x(t)}{1-\alpha}\right)k(t), \tag{C.2}$$

$$\dot{\chi}_{jc}(t) = (\sigma_c - 1)\left(\gamma_j - \bar{\gamma}_c(t)\right)\chi_{jc}(t), \tag{C.3}$$

$$\dot{\chi}_{jx}(t) = (\sigma_x - 1)\left(\gamma_j - \bar{\gamma}_x(t)\right)\chi_{jx}(t), \tag{C.4}$$

where $j = a, m, s$. Here $\bar{\gamma}_c(t) = \chi_{ac}(t)\gamma_a + \chi_{mc}(t)\gamma_m + (1 - \chi_{ac}(t) - \chi_{mc}(t))\gamma_s$ denotes the consumption expenditure-weighted average of sectoral productivity growth, and $\gamma_x(t)$ denotes its investment analog.

To a first order, the effect of structural change on growth is summarized by its effect on the weighted averages $\bar{\gamma}_c(t)$ and $\bar{\gamma}_x(t)$. The last two terms in the log-linearized Euler equation, equation (C.1), capture the two channels through which structural change affects the growth of consumption. The first term corresponds to the effect of structural change on the growth rate of the price of consumption relative to investment, which is given by the difference in the weighted average growth rate of productivity. In the log case ($\theta = 1$) this terms disappears. The second channel is the standard normalizing term that guarantees that the system is (asymptotically) stationary. This term is constant in a Neoclassical growth model, but here it is time-varying for any finite time t, and it also appears in the log-linearized law of motion of capital in equation (C.2).

Equations (C.3) and (C.4) show again that in this benchmark model, structural change, as measured by the evolution of consumption and investment expenditure shares, is independent of the Neoclassical dynamics of capital accumulation.

As discussed earlier, structural change precludes the existence of a BGP, apart from the asymptotic ones. To understand the importance of this departure, it is useful to characterize the speed of convergence of the system to the asymptotic BGPs and to determine whether the dynamics of $\bar{\gamma}_c(t)$ and $\bar{\gamma}_x(t)$ are long-lasting compared with the relatively fast Neoclassical dynamics (King and Rebelo, 1993).

Toward this end, the following proposition characterizes the eigenvalues of the system in (C.1)-(C.4), which allows us to compute the speed of convergence of the economy to the asymptotic BGPs.[2]

Proposition C.1. *The eigenvalues of the system in (C.1)-(C.4) are the solution to the following characteristic polynomial:*

$$\begin{aligned} &\overbrace{\begin{vmatrix} \alpha\left(\bar{k}_\infty\right)^{\alpha-1}-\delta-\frac{\gamma_x+\gamma_s}{1-\alpha}-\lambda & -1 \\ \bar{c}_\infty\frac{\alpha(\alpha-1)}{\theta}\left(\bar{k}_\infty\right)^{\alpha-2} & -\lambda \end{vmatrix}}^{\text{Neoclassical eigenvalues}} \\ &\times\left[\left(\sigma_c-1\right)\left(\gamma_a-\gamma_s\right)-\lambda\right]\left[\left(\sigma_c-1\right)\left(\gamma_m-\gamma_s\right)-\lambda\right] \\ &\times\underbrace{\left[\left(\sigma_x-1\right)\left(\gamma_a-\gamma_s\right)-\lambda\right]\left[\left(\sigma_x-1\right)\left(\gamma_m-\gamma_s\right)-\lambda\right]}_{\text{Structural change eigenvalues}}=0 \end{aligned} \tag{C.5}$$

where $\bar{k}_\infty$ and $\bar{c}_\infty$ are the normalized steady-state levels of consumption and capital.

Proof. Log-linearizing the system in (C.1)-(C.4), we obtain:

$$\begin{bmatrix} \dot{k}(t) \\ \dot{c}(t) \\ \dot{\chi}_{ac}(t) \\ \dot{\chi}_{mc}(t) \\ \dot{\chi}_{ax}(t) \\ \dot{\chi}_{mx}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} B_{kk} & -1 & 0 & 0 & B_{k\chi_{ax}} & B_{k\chi_{mx}} \\ B_{ck} & 0 & B_{c\chi_{ac}} & B_{c\chi_{mc}} & B_{c\chi_{ax}} & B_{c\chi_{mx}} \\ 0 & 0 & B_{\chi_{ac}\chi_{ac}} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{\chi_{mc}\chi_{mc}} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{\chi_{ax}\chi_{ax}} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{\chi_{mx}\chi_{mx}} \end{bmatrix}}_{B} \begin{bmatrix} k(t)-\bar{k}_\infty \\ c(t)-\bar{c}_\infty \\ \chi_{ac}(t) \\ \chi_{mc}(t) \\ \chi_{ax}(t) \\ \chi_{mx}(t) \end{bmatrix}$$

where $B_{kk}=\alpha\left(\bar{k}_\infty\right)^{\alpha-1}-\delta-\frac{\gamma_x}{1-\alpha}-\frac{\gamma_3}{1-\alpha}$, $B_{k\chi_{ax}}=-\frac{\gamma_a-\gamma_s}{1-\alpha}\bar{k}_\infty$, $B_{k\chi_{mx}}=-\frac{\gamma_m-\gamma_s}{1-\alpha}\bar{k}_\infty$, $B_{ck}=\bar{c}_\infty\frac{\alpha(\alpha-1)}{\theta}\left(\bar{k}_\infty\right)^{\alpha-2}$, $B_{c\chi_{ac}}=\bar{c}_\infty\frac{1-\theta}{\theta}\left(\gamma_a-\gamma_s\right)$, $B_{c\chi_{mc}}=\bar{c}_\infty\frac{1-\theta}{\theta}\left(\gamma_m-\gamma_s\right)$, $B_{c\chi_{ax}}=\bar{c}_\infty\frac{\alpha(1-\theta)-1}{\theta(1-\alpha)}\left(\gamma_a-\gamma_s\right)$, $B_{c\chi_{mx}}=\bar{c}_\infty\frac{\alpha(1-\theta)-1}{\theta(1-\alpha)}\left(\gamma_m-\gamma_s\right)$, $B_{\chi_{ac}\chi_{ac}}=\left(\sigma_c-1\right)\left(\gamma_a-\gamma_s\right)$, $B_{\chi_{mc}\chi_{mc}}=\left(\sigma_c-1\right)\left(\gamma_m-\gamma_s\right)$, $B_{\chi_{ax}\chi_{ax}}=\left(\sigma_x-1\right)\left(\gamma_a-\gamma_s\right)$,$B_{\chi_{mx}\chi_{mx}}=\left(\sigma_x-1\right)\left(\gamma_m-\gamma_s\right)$, and the remaining entries equal zero. A standard factorization of the characteristic polynomial of matrix B gives (C.5). □

Each term of the characteristic polynomial (C.5) defines eigenvalues of the system. There is a sharp separation between two sets of eigenvalues. As labeled, the first term defines a pair of eigenvalues that correspond to those in the standard Neoclassical one-sector growth model. The negative eigenvalue is associated with the stable path, and the positive, with the unstable. The remaining four terms define four eigenvalues, all negative. Among the five negative (stable) eigenvalues, the smallest in absolute value is the dominant eigenvalue because it governs the system dynamics asymptotically.

[2] Acemoglu and Guerrieri (2008) provide a related characterization in the proof of their Theorem 2.

Importantly, for quantitatively relevant values of the sectoral productivity differentials, the eigenvalues associated with structural change are smaller than those governing Neoclassical dynamics. Equivalently, the half-lives of the structural change dynamics are long relative to those of the Neoclassical transitional dynamics. For instance, if we set $\sigma_x = \sigma_c = 0$ (implying the fastest possible convergence from the structural change eigenvalues), $\gamma_a = 0.05$, $\gamma_m = 0.02$, and $\gamma_s = 0.01$, the dominant eigenvalue governing the local dynamics of structural change implies a half-life of 70 years. This is an order of magnitude larger than the half-life of the Neoclassical transitional dynamics implied by the negative Neoclassical eigenvalue.[3]

D Proof of Proposition 3

We present a constructive proof of Proposition 3. We remind the reader of the notation introduced in the main text, $c = \tilde{C}/\mathcal{A}_x^{1/(1-\alpha)}$, $k = K/\mathcal{A}_{xt}^{1/(1-\alpha)}$, $\tilde{C} = P_c C/P_x$. We also define $g_x(t) = \frac{\dot{\mathcal{A}}_x}{\mathcal{A}_x}$ and $f(t) = \frac{\dot{P}_x}{P_x} - \frac{\dot{P}_c}{P_c}$. Note that time dependence for g_x and f comes only through the evolution of sectoral TFPs, $A_i(t)$, $i = \{a, m, s\}$.[4] The dynamic system given by the Euler equation and the law of motion for capital in this notation are

$$\frac{\dot{k}}{k} = k^{\alpha-1} - \frac{c}{k} - \left(\delta + \frac{g_x(t)}{1-\alpha}\right), \tag{D.1}$$

$$\theta\frac{\dot{c}}{c} = \alpha k^{\alpha-1} - \delta - \rho + (1-\theta) f(t) - \frac{\theta}{1-\alpha} g_x(t). \tag{D.2}$$

Asymptotic Behavior: Balanced Growth Paths We begin by discussing the asymptotic behavior of the system. Note that the dynamic system becomes autonomous in the limit. Since

$$\lim_{t\to\pm\infty} f(t) = -\gamma_x, \tag{D.3}$$

$$\lim_{t\to\infty} g_x(t) = \gamma_x + \gamma_s, \tag{D.4}$$

$$\lim_{t\to-\infty} g_x(t) = \gamma_x + \gamma_a, \tag{D.5}$$

we have that the dynamic system does not directly depend on time as $t \to \pm\infty$. In this case, we have that the dynamic system has a steady state (BGP) characterized

[3] See King and Rebelo (1993) for a thorough discussion on the speed of convergence of the Neoclassical growth model.

[4] For f the factor prices cancel out since all sectors have the same factor intensities and the consumption and investment aggregators are constant returns to scale.

by the system of equations

$$
\begin{aligned}
0 =& k^{\alpha-1} - \frac{c}{k} - \left(\delta + \frac{\gamma_x + \gamma_i}{1-\alpha}\right), && \text{(D.6)}\\
0 =& \alpha k^{\alpha-1} - \delta - \rho - (1-\theta)\,\gamma_x - \frac{\theta}{1-\alpha}(\gamma_x+\gamma_i). && \text{(D.7)}
\end{aligned}
$$

Rearranging the second equation and then substituting in the first one we find that

$$
\begin{aligned}
\bar{k}_{\pm\infty} =& \left(\frac{\alpha}{\delta+\rho+(1-\theta)\,\gamma_x + \frac{\theta}{1-\alpha}(\gamma_x+\gamma_i)}\right)^{\frac{1}{1-\alpha}} && \text{(D.8)}\\
\left(\frac{\bar{c}}{\bar{k}}\right)_{\pm\infty} =& \frac{\delta+\rho+(1-\theta)\,\gamma_x + \frac{\theta}{1-\alpha}(\gamma_x+\gamma_i)}{\alpha} - \delta - \frac{\gamma_x+\gamma_i}{1-\alpha}, && \text{(D.9)}
\end{aligned}
$$

where i denotes the appropriate limiting sector. Since capital and consumption cannot be negative, we assume that the parameters above are such that this condition is satisfied. In particular, we note that this imposes the constraint

$$
\frac{\delta+\rho+(1-\theta)\,\gamma_x + \frac{\theta}{1-\alpha}(\gamma_x+\gamma_i)}{\alpha} - \delta - \frac{\gamma_x+\gamma_i}{1-\alpha} > 0. \tag{D.10}
$$

For the case $\alpha = \theta$ (that we use below) we have that

$$
\left(\frac{\bar{c}}{\bar{k}}\right)_{\pm\infty} = \frac{1-\alpha}{\alpha}\left(\delta + \gamma_x + \frac{\rho}{1-\alpha}\right) > 0. \tag{D.11}
$$

In what follows, we require that the dynamic system defined over $t \in (\underline{t}, \bar{t})$ converges to these asymptotic paths as time goes to $\pm\infty$. Proposition A.1 ensures that this limit is well defined and unique.

Model Dynamics: Computing the STraP We proceed making use of the assumption that $\alpha = \theta$. This allows us to decouple the dynamical system and first solve for $z = \frac{c}{k}$. Subtracting the law of motion for capital from the Euler equation, we have that

$$
\frac{\dot{z}}{z} = z + \delta - \frac{\delta + \rho - (1-\alpha) f(t)}{\alpha}. \tag{D.12}
$$

This differential equation to a class known as Bernoulli ordinary differential equations. To solve it, we proceed with a change of variables. Define $v = 1/z$, (note that this implies that $\dot{v} = -z^{-2}\dot{z}$) and the equation becomes a first-order linear ordinary differential equation

$$
\dot{v} - \left(\frac{\delta+\rho}{\alpha} - \delta - \frac{1-\alpha}{\alpha} f\left(t\right)\right) v + 1 = 0. \tag{D.13}
$$

This equation can be solved using the integrating factor $\lambda(t) = e^{-\int_{\underline{t}}^{t}\left[-\frac{1-\alpha}{\alpha}f(s)+\left(\frac{\delta+\rho}{\alpha}-\delta\right)\right]ds}$. The previous differential equation can be written as

$$\frac{d}{dt}\left(\lambda(t)v\right) = -\lambda(t). \tag{D.14}$$

Using that $\lambda(\underline{t}) = 1$, we can use the Fundamental Theorem of Calculus to obtain

$$\lambda(t)v(t) - v(\underline{t}) = -\int_{\underline{t}}^{t}\lambda(s)ds. \tag{D.15}$$

It is also useful to introduce notation for the integral appearing in the integrating factor, $F(t) - F(\underline{t}) \equiv \int_{\underline{t}}^{t}\left[-\frac{1-\alpha}{\alpha}f(s) + \left(\frac{\delta+\rho}{\alpha} - \delta\right)\right]ds$, so that $\lambda(\underline{t}) = e^{-(F(t)-F(\underline{t}))}$. Equation (D.15) can be rewritten as

$$e^{-F(t)}v(t) - e^{-F(\underline{t})}v(\underline{t}) = -\int_{\underline{t}}^{t}e^{-F(s)}ds. \tag{D.16}$$

Taking the limit as $t \to \infty$ and using that the system converges to the BGP, we can obtain the expression for the unique initial value $v(\underline{t})$ converging to the asymptotic BGP. Using that $\lim_{t\to\infty} v(t)e^{-F(t)} = 0$ (since v converges to a constant and $e^{F(t)}$ converges to zero), we obtain that

$$v(\underline{t}) = \frac{\int_{\underline{t}}^{\infty}e^{-F(s)}(s)\,ds}{e^{-F(\underline{t})}}. \tag{D.17}$$

In terms of the original variables, we can express this result as c and k being related by a ray through the origin:

$$c(t) = \frac{e^{-F(t)}}{\int_{t}^{\infty}e^{-F(s)}(s)\,ds}k(t) \equiv M(t)k(t), \tag{D.18}$$

where we have simplified the notation from $\underline{t}$ to t. We can readily verify using L'Hôpital's rule that as $t \to \infty$ the slope of this ray becomes constant and equal to

$$\lim_{t\to\infty} M(t) = \left\{\frac{0}{0}\right\} = \lim_{t\to\infty}\frac{e^{-F(t)}\left(-\frac{1-\alpha}{\alpha}f(s) + \left(\frac{\delta+\rho}{\alpha} - \delta\right)\right)}{e^{-F(t)}} \tag{D.19}$$

$$= \frac{1-\alpha}{\alpha}\gamma_x + \frac{\delta+\rho}{\alpha} - \delta \tag{D.20}$$

This last derivation proves Equation (20) in the main text. Also, note that Equation (D.18) shows that, by construction, $M(t)$ is a continuous function.

Once we have the policy function (D.18), we can substitute it in the law of motion for capital to obtain the solution path for capital. Imposing the boundary condition that it converges to the initial BGP as $t \to -\infty$ we then obtain the STraP. Substituting the policy function in the law of motion for capital, we have that

$$\frac{\dot{k}}{k} = k^{\alpha-1} - M(t) - \delta - \frac{g_x(t)}{1-\alpha}. \tag{D.21}$$

Defining $x = k^{1-\alpha}$ and noting that $\dot{x} = (1-\alpha) k^{-\alpha}\dot{k}$ we can rewrite the previous expression as a Bernoulli equation:

$$\dot{x} = 1 - \alpha - (1-\alpha)\left[M(t) + \frac{g_x(t)}{1-\alpha} + \delta\right] x. \tag{D.22}$$

Using the integrating factor $\mu(t) = e^{(1-\alpha)\int_{\underline{t}}^{t}\left[M(s)+\frac{g_x(s)}{1-\alpha}+\delta\right]ds}$, we have that

$$\begin{aligned} \frac{\partial}{\partial t}\left[x(t)\mu(t)\right] &= (1-\alpha)\mu(t), && \text{(D.23)}\\ x(t)\mu(t) - x(\underline{t})\mu(\underline{t}) &= (1-\alpha)\int_{\underline{t}}^{t}\mu(s)\,ds. && \text{(D.24)} \end{aligned}$$

Note that, by construction, $\mu(t)$ is continuous and increasing. Taking the limit as $\underline{t} \to -\infty$ (and using that $M(t) > 0$ and $g_x(t)$ converges to a positive number while x converges to a finite number given by the initial BGP), we find

$$\begin{aligned} x(t) &= (1-\alpha)\frac{\int_{-\infty}^{t}\mu(s)\,ds}{\mu(t)}, && \text{(D.25)}\\ k(t) &= \left[(1-\alpha)\frac{\int_{-\infty}^{t}\mu(s)\,ds}{\mu(t)}\right]^{\frac{1}{1-\alpha}}. && \text{(D.26)} \end{aligned}$$

Equation (D.26) provides the general expression for the solution of the STraP path for capital. Then, we can substitute back into the policy function (D.18) to obtain the path of consumption.

Model dynamics given an initial condition for captial k_0 To finalize the proof, consider the problem of characterizing the competitive equilibrium given an initial level of capital k_0 at time $t = 0$. In this case, we can follow the steps of the previous derivation and first characterize the path for z and then k going forward. Indeed, Equations (D.18) and (D.24) still hold in this case. The only difference relative to the previous derivation after (D.24) is that now the initial condition at $\underline{t} = 0$ is given, and we do not need to take the limit backwards to characterize the solution. Thus, we can use directly Equation (D.24) to characterize the solution,

$$\begin{aligned}
x(t, x_0) &= x_0 \frac{\mu(t_0)}{\mu(t)} + (1-\alpha) \frac{\int_{t_0}^{t} \mu(s)\, ds}{\mu(t)}, & \text{(D.27)} \\
&= x_0 \frac{\mu(t_0)}{\mu(t)} + (1-\alpha) \frac{\int_{-\infty}^{t} \mu(s)\, ds - \int_{-\infty}^{t_0} \mu(s)\, ds}{\mu(t) - \mu(t_0)} \frac{\mu(t) - \mu(t_0)}{\mu(t)}, & \text{(D.28)} \\
&= x_0 \frac{\mu(t_0)}{\mu(t)} + \left[x(t) \frac{\mu(t)}{\mu(t) - \mu(t_0)} - x(t_0) \frac{\mu(t_0)}{\mu(t) - \mu(t_0)} \right] \frac{\mu(t) - \mu(t_0)}{\mu(t)}.
\end{aligned}$$

Expressing this results in terms of the capital stock, we obtain

$$\begin{aligned}
(t, k_0) &= \left\{ k_0^{1-\alpha} \frac{\mu(t_0)}{\mu(t)} + \left[k(t)^{1-\alpha} \frac{\mu(t)}{\mu(t) - \mu(t_0)} - k(t_0)^{1-\alpha} \frac{\mu(t_0)}{\mu(t) - \mu(t_0)} \right] \frac{\mu(t) - \mu(t_0)}{\mu(t)} \right\}^{\frac{1}{1-\alpha}}, \\
&= k(t, k_0) = \left\{ \left[k_0^{1-\alpha} - k(t_0)^{1-\alpha} \right] \frac{\mu(t_0)}{\mu(t)} + k(t)^{1-\alpha} \right\}^{\frac{1}{1-\alpha}}. \qquad \text{(D.29)}
\end{aligned}$$

This last expression corresponds to the statement of the law of motion (18) stated in Proposition A.1, completing the proof.

E Computing the STraP

Although Theorem 1 ensures that a unique STraP exists, an issue of practical relevance is how to solve for the STraP. For its computation, we return to the more specific model in Section 4 and move to discrete time. We maintain the same growth notation, but we use the discrete analogs, e.g., $A_{x,t+1}/A_{x,t} = 1 + \gamma_{x,t}$, and the discount rate, ρ, is replaced by the discount factor, β. The computational algorithm we present is a double-recursive shooting algorithm. We recursively shoot both forwards and backwards. Again, we normalize values by the effective investment productivity, $\mathcal{A}_{x,t}^{1/(1-\alpha)}$, and continue denoting normalized variables using lowercase letters. In an inner loop, we shoot forward, solving for a time 0 value of consumption expenditures that asymptotically leads to the services BGP. In an outer loop, we shoot backwards and solve for a time 0 value of capital that asymptotically leads to the agriculture BGP. In practice, it is quite difficult to shoot toward an asymptotic BGP, which requires more precision at the initial value of consumption expenditure, $\tilde{c}_0$, than is computationally practical. Instead, we find a reasonably precise initial $\tilde{c}_0$, but we allow for small adjustments midpath that keep the overall path following the ideal path with a high level of precision. A useful analogy is a hypothetical launch of a rocket toward a planet in another solar system, where over time small deviations from the ideal launch angle could compound and require small retro-rocket adjustments to keep the rocket on target.

To make our computation, we assign initial values for $A_{x,0}$ and $A_{j,0}$ for $j = a, m, s$, and, for the sake of convenience, we now declare investment as the numeraire. Given these, the steps of the algorithm are as follows:

1. **Define initial bounds for k_0.** We solve for k_0 using the bisection method, which requires an upper and lower bound. Clearly, $\bar{k}_{\infty}$ and $\bar{k}_{-\infty}$ are candidate upper and lower bounds, but one can use the fact that the relative price of investment is increasing to bound it more tightly. One can solve for a pseudo-BGP level of capital, which is the normalized level of capital that would result if the effective productivities $\mathcal{A}_{x,t}$ and $\mathcal{A}_{c,t}$ grew at their initial rates perpetually:

$$k_{pseudoBGP} = \left[\frac{\alpha\beta\mathcal{A}_{x,0}}{\left(1 + \frac{\mathcal{A}_{x,1}}{\mathcal{A}_{x,0}}\right)^{1+\frac{\alpha\theta}{(1-\alpha)}} \left(1 + \frac{\mathcal{A}_{c,1}}{\mathcal{A}_{c,0}}\right)^{\theta/(1-\alpha)} - \beta\left(1 - \delta\right)} \right]^{1/(1-\alpha)}$$

   A lower bound that is sufficiently low can then be chosen. In our simulations we chose this as $0.9 * k_{pseudoBGP}$.

2. **Choose a trial value for k_0.** Using the bisection method, we choose the midpoint of the two bounds.

3. **Define initial bounds for $\tilde{c}_0$.** Again, we use the bisection method and choose upper and lower bounds. Consumption expenditures are naturally bounded between 0 and y_0, but we choose tighter bounds: $1.2 * \tilde{c}_{pseudoBGP}$ and 10 percent of initial output, $0.1 * k_0^{\alpha}$.

4. **Choose a trial value for $\tilde{c}_0$.** We choose the midpoint between the upper and lower bound.

5. **Shoot forward toward $\bar{k}_{\infty}$.** In principle, we would consider it a successful shot if we get sufficiently close to a stable value of capital close to the target, i.e., k_t within a tolerance of $\bar{k}_{\infty}$ and k_{t+1}/k_t within a tolerance of one. In this case, we would skip to the backward shooting in Step 7. However, in practice, we also stop the shooting attempt at any point in which either $\tilde{c}_{t+1} < \tilde{c}_t$ or $k_{t+1} < k_t$. (Define this point of divergence as t^*.) If the former, assign the $\tilde{c}_0$ as the new lower bound; if the latter assign $\tilde{c}_0$ as the new upper bound. Check if the bounds on $\tilde{c}_0$ are sufficiently tight. If not, return to Step 4.

6. **Update t_0 to shoot recursively.** Regardless of the precision of $\tilde{c}_0$, we have found a point of divergence, t^*. Our recursive approach is to back up from this point to some $t_n < t^*$. Practically, we choose t_n as the nearest period to $0.95 * t^*$. We then return to Step 3 at this new t_0. (Note that one need not shoot forward

until convergence to the asymptotic BGP, i.e., within a tolerance of $\bar{k}_{\infty}$, but one can stop shooting whenever the desired simulation period is finished.)

7. **Shoot backward (i.e., $t \to -\infty$) from k_0 and $\tilde{c}_0$ toward $\bar{k}_{-\infty}$.** Here we iterate backwards using the Euler equation and the laws of motion for capital and technology. Again, we consider it a successful shot if we get sufficiently close to a stable value of capital close to the target, i.e., k_t within a tolerance of $\bar{k}_{-\infty}$ and k_t/k_{t-1} within a tolerance of one. In this case, we are finished. However, we also stop the shooting attempt if capital diverges too strongly at any point. If capital becomes too large (we choose $k_{t-1} > \bar{k}_{\infty}$), we update the upper bound on k_0. If capital gets too small (we choose $k_{t-1} < 0.01$), we update the lower bound. We then return to Step 2.

The stopping procedure in the last step further illustrates the fact that capital is only backwards stable along the STraP.

F Data Construction

The paper utilizes three sets of data, which we discuss in turn.

F.1 U.S. Data

These data are used to yield the calibrations in Sections 4 and 5.

We calibrate the aggregator weights, ω_{cj} and ω_{xj}, $j = a, m, s$, to match the time series average shares of each sector in consumption and investment expenditure. We use the input-output tables to yield the sectoral composition of consumption and investment following closely the procedure described in Herrendorf et al. (2013). To do this, we combined the Make and Use tables for the years 1947-2017 produced by the Bureau of Economic Analysis (BEA) (U.S. Bureau of Economic Analysis, 2016, 2019b).[5] The input-output information from individual sectors is aggregated into our three broad aggregate sectors by simply adding the corresponding row and columns.[6]

To construct the TFP for our three broad sectors, we use data on the real output of each sector, the value added of each sector, and the aggregate capital stock.

[5]The Make table refers to the table describing the make of Commodities by Industries, before redefinitions, while the Use table refers to the table given the use of commodities by industries, before redefinitions, using producers' prices.

[6]In particular, we include agriculture, forestry, fishing, and hunting in the broad agriculture sector; mining, utilities, construction, and manufacturing in the broad manufacturing sector; and wholesale trade, retail trade, transportation and warehousing, information, finance, insurance, real estate, rental, and leasing, professional and business services, educational services, health care, and social assistance, arts, entertainment, recreation, accommodation, and food services, other services, except government, and government into the broad service sector.

The data on the real output of each sector is constructed from the Industry Economic Accounts produced by the BEA (U.S. Bureau of Economic Analysis, 2017, 2019a). In particular, we use data on the value added VA_{lt} and the chain-type price indexes P_{lt}, for disaggregated industries for the years 1947-2018. To do this combines the historical GDP by Industry data for 1947-1997 with the more recent information on the GDP by industry data for 1997-2018. We obtain quantity index for the broad aggregate sectors that we use by aggregating the individual data using the Fisher chain-weighted formula:

$$Q_{j,t} = \left[\frac{\sum_l \frac{P_{lt-1}}{P_{lt}} \mathrm{VA}_{lt}}{\sum_l \mathrm{VA}_{lt-1}} \frac{\sum_l \mathrm{VA}_{lt-1}}{\sum_l \frac{P_{lt}}{P_{lt-1}} \mathrm{VA}_{lt-1}} \right]^{\frac{1}{2}} Q_{j,t-1}.$$

The capital stock series K_t is calculated using the perpetual inventory method. We start with the value of the current-cost net stock of private fixed assets in 1947 as reported by the BEA (U.S. Bureau of Economic Analysis, 2018a), expressed in 2000 prices using the price index for private fixed investment (U.S. Bureau of Economic Analysis, 2019c). To calculate the undepreciated capital we calibrate the depreciation rate δ to match the average depreciation rate of private fixed assets over the period 1947-2017. The depreciation rate of private fixed assets is given by the ratio of the current-cost depreciation of private fixed assets to the current-cost net stock of private fixed assets (U.S. Bureau of Economic Analysis, 2018b). We add to the undepreciated capital stock the gross domestic investment (U.S. Bureau of Economic Analysis, 2018c), expressed in 2000 prices using the price index for private fixed investment.

Given the data on the quantity index for an individual sector Q_{jt}, the value added share share of this sector, $\mathrm{va}_{jt} = \mathrm{VA}_{jt} / \sum_{j'} \mathrm{VA}_{j't}$, and the aggregate capital stock K_t, the TFP of sector j, inclusive of the contribution of the changes in the aggregate labor input, is given by:

$$A_{jt} = \frac{Q_{jt}}{\mathrm{va}_{jt} \cdot K_t^{\alpha}}.$$

The key assumption to obtain this expression is that the capital can be reallocated without frictions across sectors and that all sectors have the same factor intensities α.

The neutral TFP term affecting the investment aggregator A_{xt} is calibrated using data on the chain-type price indexes of sectoral value added P_{jt} and the Price Index for Private fixed investment P_{xt} using the following expression:

$$A_{xt} = \left[\sum_{j=a,m,s} \omega_{xj} \left(\frac{P_{jt}}{P_{xt}} \right)^{1-\sigma_x} \right]^{\frac{1}{1-\sigma_x}}.$$

Finally, the discount factor is chosen so that the average return to capital in the model between 1950-2000 matches the after tax return to business capital calculated by Gomme et al. (2011).

F.2 Penn World Tables 9.1

The Penn World Tables (PWT) data are used to yield calibration targets for Section 4.5 and for data plotted in Figures 5-7. We detail the sample selection and construction of variables here.

We select the sample based on two criteria. The first criterion is the level of accuracy of the capital stock. Since PWT uses a perpetual inventory method to construct capital, the initial capital is simply assumed and can arbitrarily impact the series during early years as explained in Inklaar et al. (2019). To eliminate this issue, we obtained the additional file "pwt91_services.dta", which contains detailed and disaggregate data on capital and capital services including indicators of the type of series, and included only those series for which the initial capital stock could be constructed from historical data ("t_type"="pim", 38 countries) or those observations sufficiently past the point where initial conditions influence capital stocks ("t_type"="t_star", 92 countries but shorter time series).[7] The second criterion, as noted in the paper, is the level of income because the set of countries thins out considerably as incomes become either very high or very low. We therefore constrain the sample to country-year observations with log real income per capita between 7 (roughly $1100 in 2011 USD) and 10.7464 (roughly $46,500), the U.S. income per capita in 2000. The sample amounts to 126 countries, and 4191 country-year observations.

We plot five different variables in addition to log real income per capita.

- **Real income per capita**: In the model, the population and number of workers are equal, and output and expenditures are also equal. In the real world, however, these differ. Given the importance of nonhomotheticities, we focus on real PPP expenditure income per capita as our measure of development, which we calculate as "rgdpe"/"pop".

- **Capital-output ratio**: Because we use panel variation, we construct the ratio using current PPPs (in 2011 US$) as "cn"/"cgdp".

- **Investment rate**: We compare this to the current-value investment rate in the model ($p_x X/Y$), so we construct the analog in the data. The PWT report gross capital formation as a share or real output ("cgdpo"), "csh_i", but they do at PPPs rather than current-value shares. We therefore adjust to current value shares using the PPP prices of investment ("pl_i") and output("p_gdpo"). The

[7] Our model simulations use chain-weighted growth measures. Although the capital-output ratios are real, the timing of the initial value amounts to a critical choice of units. We follow the PWT and use current-value capital-output ratio in 1950 (see (Inklaar et al., 2019), p.43), then using chain-weighted growth in output to update values from that point.

investment rate is thus calculated as "csh_i"*"pl_i"/"p_gdpo". This is identical to the national accounts investment rate.

- **Relative price of investment**: We construct this using the PPP for investment ("pl_i") and consumption("pl_i") as: "pl_i"/"pl_c".

- **Interest rate**: The PWT 9.1 includes a measure of the internal rate or return, but this does not correspond to our consumption-based interest rate in the model, and there is no simple adjustment that we can make. Instead, we reconstruct the interest rate from scratch following the discrete model's formula for the interest rate in the Euler equation as given in footnote 11 of the paper:

$$\left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t}\right)^{\theta} = \beta\left(1+r_t\right) = \beta\left[1-\delta+\frac{R_{t+1}}{P_{x,t+1}}\right]\left[\frac{P_{x,t+1}/P_{c,t+1}}{P_{x,t}/P_{c,t}}\right]^{1-\theta}.$$

  This requires a depreciation rate, rental rate in units of the capital good, and the growth of the relative price of investment. To back out the rental rate, we start with the detailed disaggregate capital data in the file "pwt91_ services.dta". We follow PWT and construct capital share ("capsh") as one minus labor's share ("labsh") and the share of rent to natural resources ("rntsh"). We then multiply by national-accounts output ("rgdpna")to get capital payments and divide by the national-accounts capital stock ("rnna") to get the rental rate *in units of final output*. However, we require the rental rate in units of capital, so we divide by the price of capital relative to output ("pl_n/pl_gdpo").[8] Hence, the full formula for the rental rate is $rentalrate = (1 - \text{“}labsh\text{”} - \text{“}rntsh\text{”}) * \text{“}rgdpna\text{”}/\text{“}rnna\text{”}/(\text{“}pl\_n\text{”}/\text{“}pl\_gdpo\text{”})$. Next, we need to calculate the growth rate in the relative price of investment. The formula asks for an annual growth rate, but Inklaar et al. (2019) note that these can vary strongly from year to year, so the PWT averages over several years. We therefore follow the PWT in this way and construct it average annual growth rate over q five-year window in the following way

$$relativepricegrowth = (\text{“}pl\_n\text{”}_{t+2}/\text{“}pl\_con\text{”}_{t+2}/(\text{“}pl\_n\text{”}_{t-3}/\text{“}pl\_con\text{”}_{t-3}))^{(1/5)}$$

  where we have used t to index the year. Finally, using a depreciation of 0.04, the calibrated depreciation rate in the analysis, we construct the interest rate as

$$r = (1 - 0.04 + rentalrate) * relativepricegrowth - 1.$$

[8] In the model, there is only a single type of investment good so the price of investment and the price of the capital stock are identical. However, the PWT distinguishes between different types of capital, and with a changing composition of investment, the two are distinct, i.e., "pl_n" vs. "pl_i". Equating the returns to investment across capital types, these differences in prices are offset by different depreciation rates. Since our single depreciation rate is used to depreciate the overall capital stock, the price of the overall capital stock is appropriate here. The difference between the two is small, however.

- **10-year real per worker growth**: For growth rates, we utilize real variables in the national accounts, "rgdpna". Because this is a measure of growth in productivity (rather than living standards), we focus on per worker growth rather than per capita growth. Thus output per worker is "rgdpna"/"emp". Finally, growth rates are defined both in the model and data in the forward-looking manner, i.e., the growth rate at time t is $(y_{t+10}/y_t)^{1/10} - 1$. Given the forward-looking nature of the variable, we have fewer observations for this constructed variable.

F.3 Groningen Growth and Development Centre 10-Sector Database

These data are used to for the results in Figure 9. We continue our definition of the different sectors: Agriculture (includes Agriculture, forestry, and fishing), Industry (includes Mining and quarrying; Manufacturing; Public utilities; and Construction) and Services (Wholesale and retail trade, hotels and restaurants; Transport, storage, and communication; Finance, insurance, real estate; Community, social and personal services; and Government services). The raw data include 39 countries from 1950-2010, but the value-added data series for some countries are considerably shorter. Current-value shares are constructed by dividing sectoral value added by total value added. Although these data include income and population data from the PWT, the income data are output-side GDP (cgdpo) and from the PWT 8.0 version with the benchmark year of 2005. To keep consistent, we replace these data with the expenditure-side income (cgdpe) from the PWT 9.1, dividing by population (pop) to yield per capita numbers. Once again, we focus on log real incomes per capita between 7 and 10.7464, the U.S. income per capita in 2000. We are left with 36 countries and 1496 country-year observations. The countries are Argentina, Bolivia, Brazil, Botswana, Chile, Colombia, Costa Rica, Denmark, France, Ghana, Hong Kong, Indonesia, India, Italy, Japan, Kenya, Korea, Malawi, Malaysia, Mauritius, Nigeria, Netherlands, Peru, Philippines, Senegal, Singapore, South Africa, Spain, Sweden, Thailand, Taiwan, Tanzania, United States, Venezuela, and Zambia.

References

Acemoglu, D. (2009): *Introduction to Modern Economic Growth*, Princeton University Press, Princeton University Press.

Acemoglu, D. and V. Guerrieri (2008): "Capital Deepening and Nonbalanced Economic Growth," *Journal of Political Economy*, 116, 467–498.

Comin, D., D. Lashkari, and M. Mestieri (2021): "Structural Change With Long-Run Income and Price Effects," *Econometrica*, 89, 311–374.

Gomme, P., B. Ravikumar, and P. Rupert (2011): “The Return to Capital and the Business Cycle,” *Review of Economic Dynamics*, 14, 26–278.

Herrendorf, B., R. Rogerson, and A. Valentinyi (2013): “Two Perspectives on Preferences and Structural Transformation,” *American Economic Review*, 103, 2752–89.

Hubbard, J. and B. West (1991): *Differential equations: a dynamical systems approach. Part I: ordinary differential equations*, Springer-Verlag.

Inklaar, R., P. Woltjer, and D. Gallardo Albarrán (2019): “The Composition of Capital and Cross-country Productivity Comparisons,” *International Productivity Monitor*, 34–52.

King, R. G. and S. T. Rebelo (1993): “Transitional Dynamics and Economic Growth in the Neoclassical Model,” *American Economic Review*, 83, 908–931.

U.S. Bureau of Economic Analysis (2016): “Historical Make-Use Tables, 1947-1996,” https://www.bea.gov/industry/input-output-accounts-data, release Date: February 19, 2016.

——— (2017): “GDP by Industry, Historical Data, 1947-1997,” https://www.bea.gov/data/gdp/gdp-industry, release Date: March 03, 2017.

——— (2018a): “Table 1.1. Current-Cost Net Stock of Fixed Assets and Consumer Durable Goods,” https://www.bea.gov/data/investment-fixed-assets/by-type, release Date: November 20, 2018.

——— (2018b): “Table 1.3. Current-Cost Depreciation of Fixed Assets and Consumer Durable Goods,” https://www.bea.gov/data/gdp/gross-domestic-product, release Date: November 20, 2018.

——— (2018c): “Table 5.2.5. Gross and Net Domestic Investment by Major Type,” https://www.bea.gov/data/gdp/gross-domestic-product, release Date: July 31, 2018.

——— (2019a): “GDP by Industry, 1997-2018,” https://www.bea.gov/data/gdp/gdp-industry, release Date: April 19, 2019.

——— (2019b): “Make-Use Tables, Summary Level, 1997-2017,” https://www.bea.gov/industry/input-output-accounts-data, accessed 26 September 2019.

——— (2019c): “Table 5.3.4. Price Indexes for Private Fixed Investment by Type,” https://www.bea.gov/data/gdp/gross-domestic-product, release Date: April 26, 2019.