Supplement Online Appendix to "Granular Origins of Agglomeration"

Shinnosuke Kikuchi — UCSD
Daniel G. O'Connor — Princeton

September 5, 2025

This document contains additional theoretical results, proofs, and computational details for "Granular Origins of Agglomeration". In Section 1, we spell out the details of the quantitative model with granular firms in more detail, characterize the equilibrium, and prove the main theoretical results. We present the proofs of the technical lemmas in Section 2. And we provide the computational details in Section 3. Just as in the main text, we will use $\overline{x}$ to denote the value of a variable x in the absence of any ex-post shocks and $\hat{x}$ to denote log deviations from the value.

1 Details of the Quantitative, Granular Model of Economic Geography

In this section, we lay out the details of the quantitative model and prove that the theoretical results continue to hold in this more general setting.

1.1 Environment

The country is made up of I regions, indexed by $i \in \mathcal{I} \equiv \{1, \ldots, I\}$. There is a mass ℓ of workers and a continuum of sectors $s \in [0, 1]$. The sectors produce perfectly substitutable goods but hire in distinct sectoral labor markets.

Timing. There are four periods $t \in \{-1, 0, 1, 2\}$. In period -1, workers decide where to live and then stay there for the remaining periods, determining population ℓ_i. A mass m_i of firms pay a fixed cost of the traded final good in order to enter in period 0. Each firm is then randomly assigned a sector s and gets an ex-ante productivity draw z from some known distribution.

After observing those initial productivity draws, a representative worker freely allocates her labor L_{isn} across the sectors and firms in period 1. Then, in period 2, the state of the world $\omega \in \Omega$ is revealed. This determines the short-run productivity shocks to each firm. The worker can then reallocate her labor across the firms and sectors subject to moving costs. Firms then produce and sell their goods.

Workers. The fundamental utility of living in location i, u_i is

$$u_i = \overline{u}_i c_i,$$

where $\overline{u}_i$ is the local amenities and c_i is consumption of the freely traded final good. Each worker has an idiosyncratic preference for each location ε_i, so that the utility the worker gets from living in location i is $u_i \varepsilon_i$. We assume that ε_i are distributed Fréchet with shape parameter $\theta > 0$.

Once the worker has decided on a location i, she needs to make her labor supply decisions. She is a risk-neutral representative agent, endowed with one unit of labor that she supplies to the market inelastically. In period 1, the worker freely allocates her units of labor across sectors and firms. In particular, she chooses her vector of labor supply $\boldsymbol{L_i} \equiv \{L_{isn}\}_{s,n}$ in the set of feasible labor allocations $\mathcal{L}$,

$$\boldsymbol{L_i} \in \mathcal{L} \equiv \left\{ \boldsymbol{L'_i} | \int_0^1 \sum_{n \in \mathcal{N}_{is}} L'_{isn} ds \leq 1 \right\}.$$

In period 2, the state of the world ω is revealed. This determines the ex-post productivity shocks for all firms. The worker then reallocates her labor across firms, choosing a vector of labor supply $\boldsymbol{L_i(\omega)} \equiv \{L_{isn}(\omega)\}_{s,n}$ in the set of feasible labor allocations $\mathcal{L}_\Omega(\boldsymbol{L_i})$ which depends on the worker's labor choices in period 1. The set is given by

$$\mathcal{L}_\Omega(\boldsymbol{L_i}) \equiv \left\{ \boldsymbol{L_i(\omega)} | 1 = \left(\int_0^1 L_{is}^{-\frac{1}{\nu}} L_{is}(\omega)^{\frac{1+\nu}{\nu}} ds \right)^{\frac{\nu}{1+\nu}}, \right.$$
$$\left. L_{is}(\omega) = \left(\sum_{n \in \mathcal{N}_{is}} \left(\frac{L_{isn}}{L_{is}} \right)^{-\frac{1}{\kappa}} L_{isn}(\omega)^{\frac{1+\kappa}{\kappa}} \right)^{\frac{\kappa}{1+\kappa}} \right\},$$

where $L_{is} = \sum_{n \in \mathcal{N}_{is}} L_{isn}$.

Firms. There is a continuum of potential firm entrants. To enter, a firm must pay a fixed cost $\psi_i > 0$ in terms of the freely traded final good in period 0. Those firms are then randomly assigned a sector. We denote by $\mathcal{N}_{is}$ the set of firms operating in region i sector s and $N_{is} \equiv |\mathcal{N}_{is}|$ the (finite) number of firms. We assume that firms enter according to the "ball-and-urn model" so that N_{is} is distributed Poisson with mean m_i. That is, the probability mass function for the number of firms in a sector is $m_i^N e^{-m_i}/N!$.

Firm n in sector s then gets an ex-ante productivity draw z_{isn} from a distribution $F_{iz}(\cdot)$ which we assume is continuous and regularly varying.[1] We further assume that the expected HHI of a sector is decreasing and convex in the number of firms when weighted by the average productivity of those firms.[2] This assumption rules out certain distributions for F_{iz} that imply HHI increases with more firms on average. This could happen if there is some probability that a new firm would dominate the market, as it is so much more productive than the other firms.

In period 2, each firm n gets an idiosyncratic productivity shock to firm n, $\tilde{a}_{isn}(\omega)$, a sector-wide productivity shock, $\tilde{A}_{is}(\omega)$, and produces a final good, $y_{isn}(\omega)$, according to,

$$y_{isn}(\omega) = z_{isn} a_{isn}(\omega) \ell_{isn}(\omega)^{1-\eta},$$

where $a_{isn}(\omega) \equiv \tilde{a}_{isn}(\omega)\tilde{A}_{is}(\omega)$ is the total productivity shock to firm n, $\ell_{isn}(\omega)$ is the total amount of labor firm n hires, and $\eta \in (0,1)$.

We assume that $\log \tilde{a}_{isn}(\omega)$ are iid with mean zero and a finite second moment σ_N^2. Similarly, $\log \tilde{A}_{is}(\omega)$ are iid with mean zero and a finite second moment σ_S^2. We further assume that the idiosyncratic and sector-wide shocks are independent of each other.

Market Clearing. Total expected production in location i is still

$$Y_i = \mathbb{E}\left[\int_0^1 \sum_{n \in \mathcal{N}_{is}} z_{isn} a_{isn}(\omega) \ell_{isn}(\omega)^{1-\eta} ds\right].$$

The final goods are freely traded, so the market clearing condition holds at the national level,

$$\sum_{i \in \mathcal{I}} c_i \ell_i + \psi_i m_i = \sum_{i \in \mathcal{I}} Y_i. \tag{1}$$

[1]Formally, $L : (0,\infty) \to (0,\infty)$ is regularly varying if $\lim_{x\to\infty} \frac{L(ax)}{L(x)} \in \mathbb{R}^+$ for all $a > 0$. In the main text, we will assume the ex-ante distribution is Pareto, which satisfies this condition.

[2]Formally, we assume that $\mathbb{E}\left[\frac{\sum_{n' \in \mathcal{N}} z_{isn'}^{1/\eta}}{N} \frac{\sum_{n' \in \mathcal{N}} \left(z_{isn'}^{1/\eta}\right)^2}{\left(\sum_{n' \in \mathcal{N}} z_{isn'}^{1/\eta}\right)^2} \middle| N\right]$ is decreasing in N and convex for sufficiently large N where η is defined below.

In the labor market, labor demanded needs to equal the individual labor supplied by each worker multiplied by the number of workers. However, this needs to hold for each individual firm as labor is imperfectly substitutable across firms,

$$\ell_{isn}(\omega) = L_{isn}(\omega)\ell_i, \quad \forall s, n, \omega. \tag{2}$$

1.2 Market Structure and Equilibrium

Labor Supply Decison. We will characterize the worker's decision using backward induction, starting with the labor supply decision in periods 1 and 2, and then characterizing the migration decision in period -1 in the next section.

Conditional on living in location i, workers choose their labor allocation across firms and sectors in periods 1 and 2 to maximize their expected utility, taking wages as given. We normalize the price of the final goods to 1, so workers solve the problem,

$$\boldsymbol{L_i}, \boldsymbol{L_i}(\boldsymbol{\omega}) \in \underset{\boldsymbol{L}_i' \in \mathcal{L}, \boldsymbol{L}_i'(\boldsymbol{\omega}) \in \mathcal{L}_\Omega(\boldsymbol{L}_i')}{\text{argmax}} \mathbb{E}\left[\int_0^1 \sum_{n \in \mathcal{N}_{is}} w_{isn}(\omega) L'_{isn}(\omega) ds\right], \tag{3}$$

where $w_{isn}(\omega)$ is the equilibrium wage for firm n in sector s in state of the world ω. We will denote the maximum of (3) by w_i.

Writing out the maximization problem, we get

$$\max_{\boldsymbol{L}_i', \boldsymbol{L}_i'(\boldsymbol{\omega})} \mathbb{E}\left[\int_0^1 \sum_{n \in \mathcal{N}_{is}} w_{isn}(\omega) L'_{isn}(\omega) ds\right]$$

such that

$$L_{is} = \sum_{n \in \mathcal{N}_{is}} L_{isn} \tag{λ_{is}}$$

$$1 = \int_0^1 L_{is} ds \tag{λ_i}$$

$$L_{is}(\omega)^{\frac{1+\kappa}{\kappa}} = \sum_{n \in \mathcal{N}_{is}} \left(\frac{L_{isn}}{L_{is}}\right)^{-\frac{1}{\kappa}} L_{isn}(\omega)^{\frac{1+\kappa}{\kappa}} \tag{$\lambda_{is}(\omega)$}$$

$$1 = \int_0^1 (L_{is})^{-\frac{1}{\nu}} L_{is}(\omega)^{\frac{1+\nu}{\nu}} ds. \tag{$\lambda_i(\omega)$}$$

Taking the first order conditions, we find that

$$
\begin{aligned}
w_{isn}(\omega) &= \lambda_{is}(\omega)\left(\frac{L_{isn}}{L_{is}}\right)^{-\frac{1}{\kappa}}\frac{1+\kappa}{\kappa}L_{isn}(\omega)^{\frac{1}{\kappa}} \\
\lambda_{is}(\omega) &= \lambda_{i}(\omega)\left(L_{is}\right)^{-\frac{1}{\nu}}\frac{\kappa}{1+\kappa}\frac{1+\nu}{\nu}L_{is}(\omega)^{\frac{1}{\nu}-\frac{1}{\kappa}} \\
\lambda_{is} &= \mathbb{E}\left[\lambda_{is}(\omega)\frac{1}{\kappa}\left(\frac{L_{isn}}{L_{is}}\right)^{-\frac{1}{\kappa}}L_{isn}^{-1}L_{isn}(\omega)^{\frac{1+\kappa}{\kappa}}\right] \\
\lambda_{i} &= \lambda_{is} + \mathbb{E}\left[\lambda_{i}(\omega)\frac{1}{\nu}\left(L_{is}\right)^{-\frac{1}{\nu}}L_{is}^{-1}L_{is}(\omega)^{\frac{1+\nu}{\nu}}\right] - \frac{1}{\kappa}L_{is}^{-1}\mathbb{E}\left[\lambda_{is}(\omega)L_{is}(\omega)^{\frac{1+\kappa}{\kappa}}\right]. \qquad (4)
\end{aligned}
$$

One can rewrite the short run labor supply decision in the more familiar form

$$
\frac{L_{isn}(\omega)}{L_{isn}} = \left(\frac{w_{isn}(\omega)}{w_{is}(\omega)}\right)^{\kappa}\frac{L_{is}(\omega)}{L_{is}}, \quad \frac{L_{is}(\omega)}{L_{is}} = \left(\frac{w_{is}(\omega)}{w_{i}(\omega)}\right)^{\nu},
$$

where

$$
w_{is}(\omega) \equiv \left(\sum_{n\in\mathcal{N}_{is}}\frac{L_{isn}}{L_{is}}w_{isn}(\omega)^{1+\kappa}\right)^{\frac{1}{1+\kappa}}, \quad w_{i}(\omega) \equiv \left(\int_0^1 L_{is}w_{is}(\omega)^{1+\nu}ds\right)^{\frac{1}{1+\nu}}.
$$

Migration Decision. Each worker chooses the location that maximizes their utility. Therefore, population in location i satisfies

$$
\ell_i = \int_{\mathbb{R}^{\mathbb{I}}}\mathbb{1}_{i\in\text{argmax}_{i'}\,\overline{u}_{i'}w_{i'}}dG(\varepsilon)\cdot\ell, \qquad (5)
$$

where G is the joint distribution of ε. This is a standard problem in the literature with a Fréchet distribution. It implies that $\ell_i = (u_i/u)^{\theta}\cdot\ell$, where $u = \left(\sum_i (u_i)^{\theta}\right)^{\frac{1}{\theta}}$.

Labor Demand - Competitive. We will consider three different conduct assumptions on firms after they enter. The first assumption is that they are competitive. Then each active firm maximizes profits, taking wages and prices as given,

$$
\ell_{isn}(\omega) \in \underset{\ell'}{\text{argmax}} \quad z_{isn}a_{isn}(\omega)(\ell')^{1-\eta} - w_{isn}(\omega)\ell'. \qquad (6)
$$

This implies that

$$
z_{isn}a_{isn}(\omega)\ell_{isn}(\omega)^{-\eta} = w_{isn}(\omega). \qquad (7)
$$

Labor Demand - Cournot. Under Cournot competition, the firm takes as given the labor decisions of the other firms in its own sector. We assume that the firm then takes as given the workers' other options in other sectors. In the math, that will imply that the firm will take $\lambda_i(\omega)$ and λ_i in equation (4) as given. Combining some of the first-order necessary conditions of the worker's problem, we can write the firm problem as,

$$\begin{aligned}
w_{isn}(\omega), \in \quad & \underset{w'_{isn}(\omega),\ell'_{isn}(\omega),\ell'_{isn},\ell'_{is}(\omega),\ell'_{is}}{\text{argmax}} \quad \mathbb{E}\left[z_{isn}a_{isn}(\omega)\ell'_{isn}(\omega)^{1-\eta} - w'_{isn}(\omega)\ell'_{isn}(\omega)\right] \\
\ell_{isn}(\omega), \ell_{isn}, \quad \text{s.t.} \quad & (\ell'_{is})^{-\frac{1}{\kappa}}\ell'_{is}(\omega)^{\frac{1+\kappa}{\kappa}} = \left(\ell'_{isn}\right)^{-\frac{1}{\kappa}}\ell'_{isn}(\omega)^{\frac{1+\kappa}{\kappa}} + \sum_{n'\neq n}(\ell_{isn'})^{-\frac{1}{\kappa}}\ell_{isn'}(\omega)^{\frac{1+\kappa}{\kappa}} \\
\ell_{is}(\omega), \ell_{is} \quad & \ell'_{is} = \ell'_{isn} + \sum_{n'\neq n}\ell_{isn'} \\
& w'_{isn}(\omega) = \lambda_i(\omega)\frac{1+\nu}{\nu}\left(\frac{\ell_{is}(\omega)}{\ell_{is}}\right)^{\frac{1}{\nu}-\frac{1}{\kappa}}\left(\frac{\ell_{isn}(\omega)}{\ell_{isn}}\right)^{\frac{1}{\kappa}} \\
& \lambda_i(1+\kappa)\nu = \mathbb{E}\left[\lambda_i(\omega)(1+\nu)\left(\frac{\ell_{is}(\omega)}{\ell_{is}}\right)^{\frac{1}{\nu}-\frac{1}{\kappa}}\left(\frac{\ell_{isn}(\omega)}{\ell_{isn}}\right)^{\frac{1+\kappa}{\kappa}}\right] \\
& \qquad + \mathbb{E}\left[\lambda_i(\omega)(\kappa-\nu)\left(\frac{\ell_{is}(\omega)}{\ell_{is}}\right)^{\frac{1+\nu}{\nu}}\right], \qquad (8)
\end{aligned}$$

where we have transformed the per capita variables to total amount of labor.

Taking the first order conditions and simplifying gives the following necessary conditions,

$$\begin{aligned}
(1-\eta)\mathbb{E}\left[z_{isn}a_{isn}(\omega)\ell_{isn}(\omega)^{1-\eta}\right] &= \mathbb{E}[w_{isn}(\omega)\ell_{isn}(\omega)] - \ell_{isn}\mathbb{E}\left[\Lambda_{isn}(\omega)\left(\left(\frac{\ell_{isn}(\omega)}{\ell_{isn}}\right)^{\frac{1+\kappa}{\kappa}} - \left(\frac{\ell_{is}(\omega)}{\ell_{is}}\right)^{\frac{1+\kappa}{\kappa}}\right)\right] \\
(1-\eta)z_{isn}a_{isn}(\omega)\ell_{isn}(\omega)^{1-\eta} &= \frac{1+\kappa}{\kappa}w_{isn}(\omega)\ell_{isn}(\omega) - \frac{1+\kappa}{\kappa}\Lambda_{isn}(\omega)\ell_{isn}(\omega)\left(\frac{\ell_{isn}(\omega)}{\ell_{isn}}\right)^{\frac{1}{\kappa}} \\
&\quad - \frac{\Lambda^w_{isn}}{\lambda_i(\omega)}\frac{1+\kappa}{\kappa}\nu w_{isn}(\omega)\frac{\ell_{isn}(\omega)}{\ell_{isn}} \\
\Lambda_{isn}(\omega)\ell_{is}\frac{1+\kappa}{\kappa}\left(\frac{\ell_{is}(\omega)}{\ell_{is}}\right)^{\frac{1+\kappa}{\kappa}} &= -\left(\frac{1}{\nu}-\frac{1}{\kappa}\right)\left[w_{isn}(\omega)\ell_{isn}(\omega) - \frac{\Lambda^w_{isn}}{\lambda_i(\omega)}\nu w_{isn}(\omega)\frac{\ell_{isn}(\omega)}{\ell_{isn}}\right. \\
&\quad \left. - \kappa(1+\nu)\Lambda^w_{isn}\left(\frac{\ell_{is}(\omega)}{\ell_{is}}\right)^{\frac{1+\nu}{\nu}}\right] \qquad (9)
\end{aligned}$$

Labor Demand - Bertrand. Under Bertrand competition, the firm takes as given the wage decisions of the other firms in its own sector. Just as before, we assume that the firm takes as given the workers' other options in other sectors. Then we can write the firm problem as,

$$
\begin{aligned}
w_{isn}(\omega), w_{is}(\omega) \in & \underset{w'_{isn}(\omega), w'_{is}(\omega), \ell'_{isn}(\omega), \ell'_{isn}, \ell'_{is}}{\operatorname{argmax}} \mathbb{E}\left[z_{isn} a_{isn}(\omega) \ell'_{isn}(\omega)^{1-\eta} - w'_{isn}(\omega) \ell'_{isn}(\omega)\right] \\
\ell_{isn}(\omega), \ell_{isn}, \quad & \text{s.t.} \quad \ell_{is} w'_{is}(\omega)^{1+\kappa} = \ell'_{isn} w'_{isn}(\omega)^{1+\kappa} + \sum_{n' \neq n} \ell_{isn'} w_{isn'}(\omega)^{1+\kappa} \\
\ell_{is} \quad & \qquad \ell'_{is} = \ell'_{isn} + \sum_{n' \neq n} \ell_{isn'} \\
& \qquad \frac{\ell'_{isn}(\omega)}{\ell'_{isn}} = \left(\frac{\nu}{1+\nu}\right)^{\nu} \lambda_i(\omega)^{-\nu} w'_{is}(\omega)^{\nu-\kappa} w'_{isn}(\omega)^{\kappa} \\
& \qquad \lambda_i = \frac{1}{1+\kappa} \left(\frac{\nu}{1+\nu}\right)^{\nu} \mathbb{E}\left[\lambda_i(\omega) w'_{is}(\omega)^{\nu-\kappa} w'_{isn}(\omega)^{1+\kappa}\right] \\
& \qquad\qquad + \frac{\kappa}{1+\kappa} \left(\frac{1}{\nu} - \frac{1}{\kappa}\right) \left(\frac{\nu}{1+\nu}\right)^{1+\nu} \mathbb{E}\left[\lambda_i(\omega) w'_{is}(\omega)^{1+\nu}\right].
\end{aligned}
\tag{10}
$$

Taking the first order conditions and simplifying gives the following necessary conditions,

$$
\begin{aligned}
\mathbb{E}\left[(1-\eta) z_{isn} a_{isn}(\omega) \ell_{isn}(\omega)\right] &= \mathbb{E}[w_{isn}(\omega) \ell_{isn}(\omega)] \\
&\quad - \ell_{isn} \mathbb{E}\left[\Lambda_{isn}(\omega)\left(w_{isn}(\omega)^{1+\kappa} - w_{is}(\omega)^{1+\kappa}\right)\right] \\
(1-\eta) z_{isn} a_{isn}(\omega) \ell_{isn}(\omega) &= \frac{1+\kappa}{\kappa} w_{isn}(\omega) \ell_{isn}(\omega) - \frac{1+\kappa}{\kappa} \Lambda_{isn}(\omega) \ell_{isn} w_{isn}(\omega)^{1+\kappa} \\
&\quad - \frac{\Lambda^{w}_{isn}}{\kappa} \lambda_i(\omega)^{-\nu} \left(\frac{\nu}{1+\nu}\right)^{\nu} w_{is}(\omega)^{\nu-\kappa} w_{isn}(\omega)^{1+\kappa} \\
\frac{1+\kappa}{\kappa} \Lambda_{isn}(\omega) \Big(\kappa \ell_{is} w_{is}(\omega)^{1+\kappa} + & \\
(\nu-\kappa) \ell_{isn} w_{isn}(\omega)^{1+\kappa} \Big) &= \frac{\nu-\kappa}{\kappa} w_{isn}(\omega) \ell_{isn}(\omega) \\
&\quad + \frac{1}{1+\kappa} \Lambda^{w}_{isn} \lambda(\omega)^{-\nu} \left(\frac{\nu}{1+\nu}\right)^{\nu} \frac{\kappa-\nu}{\kappa} w_{is}(\omega)^{\nu-\kappa} w_{isn}(\omega)^{1+\kappa} \\
&\quad + \Lambda^{w}_{isn} \lambda_i(\omega)^{-\nu} \frac{\kappa-\nu}{1+\kappa} \left(\frac{\nu}{1+\nu}\right)^{\nu} w_{is}(\omega)^{1+\nu}
\end{aligned}
\tag{11}
$$

Entry. Entry is the same as in the baseline model. Thus, free entry implies that average profits are equal to the cost of entry

$$
\psi_i = \frac{\mathbb{E}\left[\int_0^1 \sum_{n \in \mathcal{N}_{is}} \pi_{isn}(\omega) ds\right]}{m_i}. \tag{12}
$$

1.3 Theoretical Results for the Competitive Case

In this section, we prove the propositions of the main text in the more general case with $\kappa, \nu \in (0, \infty)$.

Proposition 1. *In any stable equilibrium, the average wage is increasing in the number of workers, i.e.* $\frac{d \log w_i}{d \log \ell_i} > 0$*, if and only if the employment weighted covariance between log firm productivity and log firm employment is increasing in m, i.e.* $\frac{\partial \log \Phi(m)}{\partial \log m}\big|_{m=m_i} > 0$.

Proof. The proof exactly follows the argument in the main text so we do not reporduce it here. □

Proposition 2. *If idiosyncratic shocks have a positive variance,* $\sigma_N^2 > 0$*, and labor is more substitutable across firms within a sector than across sectors,* $\kappa > \nu$*, the average wage is increasing in the number of workers, i.e.* $\frac{d \log w_i}{d \log \ell_i} > 0$*. Furthermore, the agglomeration benefits converge to zero as the size of the market goes to infinity, i.e.* $\frac{d \log w_i}{d \log \ell_i} \to 0$ *as* $m_i \to \infty$.

Proof. Taking a log first order approximation to the labor supply FOCs (4) after substituting out $\lambda_{is}(\omega)$ and λ_{is} implies,

$$\hat{w}_{isn}(\omega) = \hat{\lambda}(\omega) + \left(\frac{1}{\nu} - \frac{1}{\kappa}\right)\left(\hat{\ell}_{is}(\omega) - \hat{\ell}_{is}\right) + \frac{1}{\kappa}\left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{is}\right)$$
$$\hat{\bar{\lambda}}_i = \frac{1+\nu}{1+\kappa}\mathbb{E}\left[\hat{w}_{isn}(\omega)\right] + \frac{\kappa - \nu}{1+\kappa}\mathbb{E}\left[\hat{\lambda}(\omega)\right] + \frac{\kappa}{1+\kappa}\frac{1+\nu}{\nu}\mathbb{E}\left[\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn}\right],$$

where we use the fact that $\hat{L}_{isn}(\omega) = \hat{\ell}_{isn}(\omega)$.

Taking a log first order approximation to the labor constraints embedded in $\mathcal{L}$ and $\mathcal{L}_\Omega(\cdot)$ implies,

$$\hat{\ell}_{is}(\omega) = \sum_{n \in \mathcal{N}_{is}} \frac{\bar{\ell}_{isn}}{\bar{\ell}_{is}}\hat{\ell}_{isn}(\omega), \qquad 0 = \int_0^1 \frac{\bar{\ell}_{is}}{\ell_i}\hat{\ell}_{is}(\omega) ds,$$
$$\hat{\ell}_{is} = \sum_{n \in \mathcal{N}_{is}} \frac{\bar{\ell}_{isn}}{\bar{\ell}_{is}}\hat{\ell}_{isn}, \qquad 0 = \int_0^1 \frac{\bar{\ell}_{is}}{\ell_i}\hat{\ell}_{is} ds.$$

And the labor demand curve implies that $\hat{a}_{isn}(\omega) - \eta\hat{\ell}_{isn}(\omega) = \hat{w}_{isn}(\omega)$. First note that taking expectations $\mathbb{E}[\hat{a}_{isn}(\omega)] - \eta\mathbb{E}[\hat{\ell}_{isn}(\omega)] = \mathbb{E}[\hat{w}_{isn}(\omega)]$ which implies $\mathbb{E}[\hat{w}_{isn}(\omega)] = -\eta\mathbb{E}[\hat{\ell}_{isn}(\omega)]$. Therefore,

$$-\hat{\lambda}_i(\omega) = \left(\frac{1}{\nu} - \frac{1}{\kappa}\right)\left(\hat{\ell}_{is}(\omega) - \hat{\ell}_{is}\right) + \frac{1}{\kappa}\left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{is}\right) + \eta\hat{\ell}_{isn}(\omega)$$

$$
\begin{aligned}
-\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \hat{\lambda}(\omega) ds = \left(\frac{1}{\nu} - \frac{1}{\kappa}\right) \int_0^1 \frac{\overline{\ell}_{is}}{\ell_i} \left(\hat{\ell}_{is}(\omega) - \hat{\ell}_{is}\right) ds \\
+ \int_0^1 \frac{\overline{\ell}_{is}}{\ell_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \left(\frac{1}{\kappa}\left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{is}\right) + \eta \hat{\ell}_{isn}(\omega)\right) ds \\
\hat{\lambda}(\omega) = 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
\hat{\lambda}_i &= \frac{1+\nu}{1+\kappa}\mathbb{E}\left[\hat{w}_{isn}(\omega)\right] + \frac{\kappa-\nu}{1+\kappa}\mathbb{E}\left[\hat{\lambda}(\omega)\right] + \frac{\kappa}{1+\kappa}\frac{1+\nu}{\nu}\mathbb{E}\left[\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn}\right] \\
\hat{\lambda}_i &= \frac{1+\nu}{1+\kappa}\left(\frac{\kappa}{\nu} - \eta\right)\mathbb{E}\left[\hat{\ell}_{isn}(\omega)\right] - \frac{\kappa}{1+\kappa}\frac{1+\nu}{\nu}\hat{\ell}_{isn}.
\end{aligned}
$$

Again taking a weighted sum across all firms implies that $\hat{\lambda}_i = 0$. Since $\kappa/\nu > 1$ and $\eta \in (0,1)$, it then follows that $\hat{\ell}_{isn} = \mathbb{E}[\hat{\ell}_{isn}(\omega)] = \mathbb{E}[\hat{w}_{isn}(\omega)] = 0$ because otherwise it is not possible for $0 = \left(\frac{\kappa}{\nu} - \eta\right)\mathbb{E}\left[\hat{\ell}_{isn}(\omega)\right] - \frac{\kappa}{\nu}\hat{\ell}_{isn}$ and the labor constraints to hold.

Therefore, we can solve for sectoral labor

$$
\begin{aligned}
\hat{a}_{isn}(\omega) - \eta\hat{\ell}_{isn}(\omega) &= \left(\frac{1}{\nu} - \frac{1}{\kappa}\right)\hat{\ell}_{is}(\omega) + \frac{1}{\kappa}\hat{\ell}_{isn}(\omega) \\
\sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{a}_{isn}(\omega) &= \left(\frac{1}{\nu} - \frac{1}{\kappa}\right)\hat{\ell}_{is}(\omega) + \left(\frac{1}{\kappa} + \eta\right)\hat{\ell}_{is}(\omega) \\
\frac{1}{\eta + \frac{1}{\nu}} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{a}_{isn}(\omega) &= \hat{\ell}_{is}(\omega).
\end{aligned}
$$

And individual firm labor is

$$
\hat{\ell}_{isn}(\omega) = \frac{1}{\eta + \frac{1}{\kappa}}\left(\hat{a}_{isn}(\omega) - \frac{\frac{1}{\nu} - \frac{1}{\kappa}}{\eta + \frac{1}{\nu}} \sum_{n'\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn'}}{\overline{\ell}_{is}}\hat{a}_{isn'}(\omega)\right).
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}[\hat{a}_{isn}(\omega)\hat{\ell}_{isn}(\omega)] &= \mathbb{E}\left[\hat{a}_{isn}(\omega)\frac{1}{\eta + \frac{1}{\kappa}}\left(\hat{a}_{isn}(\omega) - \frac{\frac{1}{\nu} - \frac{1}{\kappa}}{\eta + \frac{1}{\nu}} \sum_{n'\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn'}}{\overline{\ell}_{is}}\hat{a}_{isn'}(\omega)\right)\right] \\
&= \frac{1}{\eta + \frac{1}{\kappa}}\left(\sigma_S^2 + \sigma_N^2\right) - \frac{\frac{1}{\nu} - \frac{1}{\kappa}}{\left(\eta + \frac{1}{\kappa}\right)\left(\eta + \frac{1}{\nu}\right)}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\sigma_N^2 + \sigma_S^2\right) \\
&= \frac{1}{\eta + \frac{1}{\nu}}\sigma_S^2 + \frac{\eta + \frac{1}{\nu} - \left(\frac{1}{\nu} - \frac{1}{\kappa}\right)\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}}{\left(\eta + \frac{1}{\kappa}\right)\left(\eta + \frac{1}{\nu}\right)}\sigma_N^2.
\end{aligned}
$$

Then we can calculate $\Phi(m)$,

$$\begin{aligned}\Phi(m) &= \mathbb{E}[a_{isn}(\omega)] + \frac{1-\eta}{2}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}\left[\hat{a}_{isn}(\omega)\hat{\ell}_{isn}(\omega)\right] \\ &= \mathbb{E}[a_{isn}(\omega)] + \frac{1-\eta}{2}\frac{1}{\eta+\frac{1}{\nu}}\sigma_S^2 + \frac{1-\eta}{2}\frac{\eta+\frac{1}{\nu} - \left(\frac{1}{\nu}-\frac{1}{\kappa}\right)\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds}{\left(\eta+\frac{1}{\kappa}\right)\left(\eta+\frac{1}{\nu}\right)}\sigma_N^2.\end{aligned}$$

Therefore,

$$\frac{\partial \log \Phi(m)}{\partial \log m} = -\frac{1-\eta}{2}\frac{\frac{1}{\nu}-\frac{1}{\kappa}}{\left(\eta+\frac{1}{\kappa}\right)\left(\eta+\frac{1}{\nu}\right)}\frac{\partial}{\partial \log m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds\right]\frac{\sigma_N^2}{\Phi(m)}$$

By Lemma 5, $\frac{\partial}{\partial \log m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds\right] < 0$. Then by Proposition 1, the first result will follow. Similarly, by Lemma 5, $\frac{\partial}{\partial \log m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds\right] \to 0$ as $m\to\infty$ so

$$\frac{d\log w_i}{d\log \ell_i} = \frac{\frac{1}{1-\eta}\frac{\partial \log \Phi(m_i)}{\partial \log m_i}}{1-\frac{1}{1-\eta}\frac{\partial \log \Phi(m_i)}{\partial \log m_i}} \to 0,$$

as $m\to\infty$. □

Proposition 3. *If idiosyncratic shocks have a positive variance, $\sigma_N^2 > 0$, the optimal policy features a subsidy on firm entry proportional to firm profits given by $\tau_i = \frac{1}{\eta}\frac{\partial \log\Phi(m)}{\partial \log m}\big|_{m=m_i}$. Furthermore, the optimal subsidy converges to zero as the size of the market goes to infinity, i.e. $\tau_i \to 0$ as $m_i \to \infty$.*

Proof. The proof exactly follows the argument in the main text so we do not reproduce it here. □

1.4 Characterizing Imperfect Competition

In this subsection, we characterize the equilibrium under Cournot and Bertrand Competition. In both cases, the expression for production given in Lemma 6 holds no matter how firms behave. Therefore, we go through and characterize what total wage payments are which then implies profits.

Taking a second order approximation to total wage payments and the firm FOCs associated with Cournot competition implies the next proposition.

Proposition 4. *If firms compete a lá Cournot, total wage compensation in location i can be written,*

$$w_i \ell_i = (1-\eta) z_i (m_i)^{\eta} (\ell_i)^{1-\eta} \left(\tilde{\Phi}(m) + \Psi^c(m) \right),$$

where $\tilde{\Phi}(m)$ *is defined as in Lemma 6 and*

$$\Psi^c(m) \equiv \frac{1+\kappa}{\kappa} \int_{\mathcal{S}} \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_n \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E} \left[\frac{\hat{\Lambda}_{isn}(\omega)}{w_i} \left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn} - \hat{\ell}_{is}(\omega) + \hat{\ell}_{is} \right) \right] ds.$$

Proof. First we note that because workers are freely mobile across all sectors in period 1, the equilibrium with no ex-post shocks and Cournot competition is the same as the equilibrium with no ex-post shocks and competitive firms. Doing a second order approximation around the point with no ex-post shocks implies

$$\begin{aligned} &\mathbb{E}\left[\int_0^1 \sum_{n \in \mathcal{N}_{is}} w_{isn}(\omega) \ell_{isn}(\omega) \right] \\ &= \overline{w}_i \overline{\ell}_i \Bigg(\int_0^1 \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\ell_i} \left(1 + \mathbb{E}[\hat{w}_{isn}(\omega)] + \mathbb{E}[\hat{\ell}_{isn}(\omega)] + \frac{1}{2} \mathbb{E}[(\hat{w}_{isn}(\omega) + \hat{\ell}_{isn}(\omega))^2] \right) \Bigg) \end{aligned}$$

We then take a log second order approximation to the firm FOCs in equation (9). However, with no shocks $\Lambda_{isn}(\omega) = 0$, so we leave that in levels. The first equation becomes

$$\begin{aligned} \mathbb{E}[\hat{a}_{isn}(\omega)] + (1-\eta)\mathbb{E}[\hat{\ell}_{isn}(\omega)] + \frac{1}{2}\mathbb{E}\left[(\hat{a}_{isn}(\omega) + (1-\eta)\hat{\ell}_{isn}(\omega))^2 \right] \\ = \mathbb{E}[\hat{w}_{isn}(\omega)] + \mathbb{E}[\hat{\ell}_{isn}(\omega)] + \frac{1}{2}\mathbb{E}\left[(\hat{w}_{isn}(\omega) + \hat{\ell}_{isn})^2 \right] \\ - \frac{1+\kappa}{\kappa} \mathbb{E}\left[\frac{\hat{\Lambda}_{isn}(\omega)}{w_i} \left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn} - \hat{\ell}_{is}(\omega) + \hat{\ell}_{is} \right) \right]. \end{aligned}$$

We also need the second order approximation to the labor constraints embedded in $\mathcal{L}$ and $\mathcal{L}_{\Omega}(\cdot)$,

$$\begin{aligned} &\hat{\ell}_{is}(\omega) + \frac{1}{2}\frac{\kappa}{1+\kappa} \left(-\frac{1}{\kappa}\hat{\ell}_{is} + \frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega) \right)^2 + \frac{1}{1+\kappa}\hat{\ell}_{is}^2 \\ &\quad = \sum_{n \in \mathcal{N}_{is}} \frac{\ell_{isn}}{\overline{\ell}_{is}} \left(\hat{\ell}_{isn}(\omega) + \frac{1}{2}\frac{\kappa}{1+\kappa} \left(-\frac{1}{\kappa}\hat{\ell}_{isn} + \frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega) \right)^2 + \frac{1}{1+\kappa}\hat{\ell}_{isn}^2 \right) \end{aligned}$$

$$0 = \int_0^1 \overline{\ell}_{is} \left(\hat{\ell}_{is}(\omega) + \frac{1}{2}\frac{\nu}{1+\nu} \left(-\frac{1}{\nu}\hat{\ell}_{is} + \frac{1+\nu}{\nu}\hat{\ell}_{is}(\omega) \right)^2 + \frac{1}{1+\nu}\hat{\ell}_{is}^2 \right).$$

So then we can write

$$\begin{aligned}
&\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \left(\mathbb{E}\left[\hat{w}_{isn}(\omega)\right] + \mathbb{E}\left[\hat{\ell}_{isn}(\omega)\right] + \frac{1}{2}\mathbb{E}[(\hat{w}_{isn}(\omega) + \hat{\ell}_{isn}(\omega))^2] \right) ds \\
&\quad = \frac{1}{2}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}\left[(\hat{a}_{isn}(\omega) + (1-\eta)\hat{\ell}_{isn}(\omega))^2 \right] ds \\
&\qquad + \frac{1+\kappa}{\kappa} \int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}\left[\frac{\hat{\Lambda}_{isn}(\omega)}{w_i} \left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn} - \hat{\ell}_{is}(\omega) + \hat{\ell}_{is} \right) \right] ds \\
&\qquad + \int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \left(\mathbb{E}[\hat{a}_{isn}(\omega)] + (1-\eta)\mathbb{E}[\hat{\ell}_{isn}(\omega)] \right) ds \\
&\quad = \mathbb{E}[\hat{a}_{isn}(\omega)] + \frac{1}{2}\mathbb{E}[\hat{a}_{isn}(\omega)^2] \\
&\qquad + (1-\eta) \int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}[\hat{a}_{isn}(\omega)\hat{\ell}_{isn}(\omega)] ds \\
&\qquad - \eta\frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_{is}}{\ell} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}[\hat{\ell}_{isn}(\omega)^2] ds \\
&\qquad - \frac{1}{\kappa}\frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_{s}}{\ell} \sum_{n\in\mathcal{N}_{s}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}\left[\left(\hat{\ell}_{sn}(\omega) - \hat{\ell}_{isn} \right)^2 \right] ds \\
&\qquad - \left(\frac{1}{\nu} - \frac{1}{\kappa} \right) \frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_{is}}{\ell} \mathbb{E}\left[\left(\hat{\ell}_{is}(\omega) - \hat{\ell}_{is} \right)^2 \right] ds \\
&\qquad + \frac{1+\kappa}{\kappa} \int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}\left[\frac{\hat{\Lambda}_{isn}(\omega)}{w_i} \left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn} - \hat{\ell}_{is}(\omega) + \hat{\ell}_{is} \right) \right] ds.
\end{aligned}$$

This completes the proof. □

Then we can do the same thing when firms are behaving Bertrand.

Proposition 5. *If firms compete a lá Bertrand, total wage compensation in location i can be written,*

$$w_i\ell_i = (1-\eta)z_i(m_i)^\eta(\ell_i)^{1-\eta}\left(\tilde{\Phi}(m) + \Psi^b(m) \right),$$

where $\tilde{\Phi}(m)$ is defined as in Lemma 6 and

$$\Psi^b(m) \equiv (1+\kappa)\,(w_i)^\kappa \int_{\mathcal{S}} \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_n \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}\left[\hat{\Lambda}_{isn}(\omega)\left(\hat{w}_{isn}(\omega) - \hat{w}_{is}(\omega) \right) \right] ds.$$

Proof. First we note that because workers are freely mobile across all sectors in period 1, the equilibrium with no ex-post shocks and Cournot competition is the same as the equilibrium with no ex-post shocks and competitive firms. Doing a second order approximation around the point with no ex-post shocks implies

$$\begin{aligned}&\mathbb{E}\left[\int_0^1 \sum_{n\in\mathcal{N}_{is}} w_{isn}(\omega)\ell_{isn}(\omega)\right]\\&\quad=\overline{w}_i\overline{\ell}_i\bigg(\int_0^1 \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\ell_i}\left(1+\mathbb{E}[\hat{w}_{isn}(\omega)]+\mathbb{E}[\hat{\ell}_{isn}(\omega)]+\frac{1}{2}\mathbb{E}[(\hat{w}_{isn}(\omega)+\hat{\ell}_{isn}(\omega))^2]\right)\bigg)\end{aligned}$$

We then take a to the firm FOCs in equation (11). However, with no shocks $\Lambda_{isn}(\omega)=0$, so we leave that in levels. The first equation becomes,

$$\begin{aligned}\mathbb{E}[\hat{a}_{isn}(\omega)]+(1-\eta)\mathbb{E}[\hat{\ell}_{isn}(\omega)]+\frac{1}{2}\mathbb{E}\left[(\hat{a}_{isn}(\omega)+(1-\eta)\hat{\ell}_{isn}(\omega))^2\right]&\\=\mathbb{E}[\hat{w}_{isn}(\omega)]+\mathbb{E}[\hat{\ell}_{isn}(\omega)]+\frac{1}{2}\mathbb{E}\left[(\hat{w}_{isn}(\omega)+\hat{\ell}_{isn})^2\right]&\\-(1+\kappa)\,(w_i)^{\kappa}\,\mathbb{E}\left[\hat{\Lambda}_{isn}(\omega)\,(\hat{w}_{isn}(\omega)-\hat{w}_{is}(\omega))\right].&\end{aligned}$$

We also need the second order approximation to the labor constraints embedded in $\mathcal{L}$ and $\mathcal{L}_{\Omega}(\cdot)$,

$$\begin{aligned}\hat{\ell}_{is}(\omega)+&\frac{1}{2}\frac{\kappa}{1+\kappa}\left(-\frac{1}{\kappa}\hat{\ell}_{is}+\frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega)\right)^2+\frac{1}{1+\kappa}\hat{\ell}_{is}^2\\&=\sum_{n\in\mathcal{N}_{is}}\frac{\ell_{isn}}{\overline{\ell}_{is}}\left(\hat{\ell}_{isn}(\omega)+\frac{1}{2}\frac{\kappa}{1+\kappa}\left(-\frac{1}{\kappa}\hat{\ell}_{isn}+\frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega)\right)^2+\frac{1}{1+\kappa}\hat{\ell}_{isn}^2\right)\\0&=\int_0^1\overline{\ell}_{is}\left(\hat{\ell}_{is}(\omega)+\frac{1}{2}\frac{\nu}{1+\nu}\left(-\frac{1}{\nu}\hat{\ell}_{is}+\frac{1+\nu}{\nu}\hat{\ell}_{is}(\omega)\right)^2+\frac{1}{1+\nu}\hat{\ell}_{is}^2\right).\end{aligned}$$

So then we can write

$$\begin{aligned}&\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\left(\mathbb{E}\left[\hat{w}_{isn}+(\omega)\right]+\mathbb{E}\left[\hat{\ell}_{isn}(\omega)\right]+\frac{1}{2}\mathbb{E}[(\hat{w}_{isn}(\omega)+\hat{\ell}_{isn}(\omega))^2]\right)ds\\&\quad=\frac{1}{2}\int_0^1\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\mathbb{E}\left[(\hat{a}_{isn}(\omega)+(1-\eta)\hat{\ell}_{isn}(\omega))^2\right]ds\\&\qquad+(1+\kappa)\,(w_i)^{\kappa}\int_{\mathcal{S}}\frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_n\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\mathbb{E}\left[\hat{\Lambda}_{isn}(\omega)\,(\hat{w}_{isn}(\omega)-\hat{w}_{is}(\omega))\right]ds\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \left(\mathbb{E}[\hat{a}_{isn}(\omega)] + (1-\eta)\mathbb{E}[\hat{\ell}_{isn}(\omega)]\right) ds \\
= {}& \mathbb{E}[\hat{a}_{isn}(\omega)] + \frac{1}{2}\mathbb{E}[\hat{a}_{isn}(\omega)^2] \\
&+ (1-\eta)\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}[\hat{a}_{isn}(\omega)\hat{\ell}_{isn}(\omega)]ds \\
&- \eta\frac{1-\eta}{2}\int_0^1 \frac{\overline{\ell}_{is}}{\ell} \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}[\hat{\ell}_{isn}(\omega)^2]ds \\
&- \frac{1}{\kappa}\frac{1-\eta}{2}\int_0^1 \frac{\overline{\ell}_{s}}{\ell} \sum_{n\in\mathcal{N}_{s}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}\left[\left(\hat{\ell}_{sn}(\omega) - \hat{\ell}_{isn}\right)^2\right] ds \\
&- \left(\frac{1}{\nu} - \frac{1}{\kappa}\right)\frac{1-\eta}{2}\int_0^1 \frac{\overline{\ell}_{is}}{\ell} \mathbb{E}\left[\left(\hat{\ell}_{is}(\omega) - \hat{\ell}_{is}\right)^2\right] ds \\
&+ (1+\kappa)\,(w_i)^{\kappa} \int_{\mathcal{S}} \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \mathbb{E}\left[\hat{\Lambda}_{isn}(\omega)\left(\hat{w}_{isn}(\omega) - \hat{w}_{is}(\omega)\right)\right] ds.
\end{aligned}
$$

This completes the proof. □

Proposition 4 and Proposition 5 gives wages completely in second order terms for Cournot competition and Bertrand competition, respectively. Therefore, following Benigno and Woodford (2003) and Benigno and Woodford (2012), we can then approximate the labor constraints and firm first order conditions with a log first order approximation. We then computationally evaluate $\tilde{\Phi}(m)$ and $\Psi(m)$ for these two different cases and use those functions in calculating the equilibrium.

2 Proofs of Technical Lemmas

In this section, we prove the Technical Lemmas used in the main text and Section 1 for the more general case with $\kappa, \nu \in (0, \infty)$. To get the results for the results in the main text, one simply needs to take the limit as $\kappa \to \infty$ and $\nu \to 0$.

Lemma 1. *If firms are competitive conditional on entry, the regional production function is* $Y_i(\ell, m) = z_i m^{\eta} \ell^{1-\eta} \Phi(m)$*, where* $z_i \equiv \mathbb{E}[z_{isn}^{1/\eta}]^{\eta}$ *and* $\Phi(m)$ *is given by,*

$$
\Phi(m) \equiv \mathbb{E}[a_{sn}(\omega)] + \frac{1-\eta}{2}\int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n\in\mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathrm{Cov}\left(\log a_{sn}(\omega), \log \ell_{sn}(\omega)\right) ds. \tag{13}
$$

Proof. By Lemma 6, the regional production can be written $Y_i(\ell, m) = z_i m^{\eta} \ell^{1-\eta} \tilde{\Phi}(m)$

where

$$\tilde{\Phi}(m) \equiv \mathbb{E}[a_{sn}(\omega)] + (1-\eta)\int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n\in\mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}[\hat{a}_{sn}(\omega)\hat{\ell}_{sn}(\omega)]ds$$
$$-\eta\frac{1-\eta}{2}\int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n\in\mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}[\hat{\ell}_{sn}(\omega)^2]ds$$
$$-\frac{1}{\kappa}\frac{1-\eta}{2}\int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n\in\mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}\left[\left(\hat{\ell}_{sn}(\omega)-\hat{\ell}_{sn}\right)^2\right]ds$$
$$-\left(\frac{1}{\nu}-\frac{1}{\kappa}\right)\frac{1-\eta}{2}\int_0^1 \frac{\overline{\ell}_s}{\ell} \mathbb{E}\left[\left(\hat{\ell}_s(\omega)-\hat{\ell}_s\right)^2\right]ds$$

Therefore, the planner is looking to maximize this second order production function subject to the labor constraints. Following Benigno and Woodford (2003) and Benigno and Woodford (2012), we can do a linear approximation to the labor constraints, embedded in the sets $\mathcal{L}$ and $\mathcal{L}_\Omega(\cdot)$,

$$\hat{\ell}_{is}(\omega) = \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{\ell}_{isn}(\omega), \qquad 0 = \int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\hat{\ell}_{is}(\omega)ds,$$
$$\hat{\ell}_{is} = \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{\ell}_{isn}, \qquad 0 = \int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\hat{\ell}_{is}ds.$$

To simplify the explication, we rewrite the maximization problem with vector notation as

$$\max_x ax - \frac{1}{2}x'bx$$

such that

$$cx = 0,$$

where x is the vector of labor supply decisions, a is the vector of productivity shocks so that

$$ax = (1-\eta)\int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n\in\mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}[\hat{a}_{sn}(\omega)\hat{\ell}_{sn}(\omega)]ds,$$

b is the self adjoint operator (i.e. symmetric matrix) representing the loss function so that

$$x'bx = \eta(1-\eta)\int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n\in\mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}[\hat{\ell}_{sn}(\omega)^2]ds$$
$$+\frac{1}{\kappa}(1-\eta)\int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n\in\mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}\left[\left(\hat{\ell}_{sn}(\omega)-\hat{\ell}_{sn}\right)^2\right]ds$$

$$+\left(\frac{1}{\nu}-\frac{1}{\kappa}\right)(1-\eta)\int_0^1 \frac{\overline{\ell}_s}{\ell}\mathbb{E}\left[\left(\hat{\ell}_s(\omega)-\hat{\ell}_s\right)^2\right]ds,$$

and c is the matrix representing the linear constraints. Forming the lagrangian we have

$$ax-\frac{1}{2}x'bx-\lambda cx.$$

Taking the FOCs we get

$$a-(bx)'-\lambda c=0.$$

Therefore, $x=b^{-1}(a-\lambda c)'=b^{-1}(a'-c'\lambda')$. Using the constraint, we can solve for λ:

$$\begin{aligned}0&=cx\\&=cb^{-1}(a'-c'\lambda')\\\lambda'&=(cb^{-1}c')^{-1}cb^{-1}a'.\end{aligned}$$

Then to complete the proof, we will show that $ax=x'bx$.

$$\begin{aligned}x'bx&=\left(b^{-1}(a'-c'(cb^{-1}c')^{-1}cb^{-1}a')\right)'bb^{-1}(a'-c'(cb^{-1}c')^{-1}cb^{-1}a')\\&=(a-ab^{-1}c'(cb^{-1}c')^{-1}c)b^{-1}(a'-c'(cb^{-1}c')^{-1}cb^{-1}a')\\&=ab^{-1}a'-ab^{-1}c'(cb^{-1}c')^{-1}cb^{-1}a'-ab^{-1}c'(cb^{-1}c')^{-1}cb^{-1}a'\\&\quad+ab^{-1}c'(cb^{-1}c')^{-1}cb^{-1}c'(cb^{-1}c')^{-1}cb^{-1}a'\\&=ab^{-1}a'-ab^{-1}c'(cb^{-1}c')^{-1}cb^{-1}a',\end{aligned}$$

where we use the fact that b and $cb^{-1}c'$ are self adjoint and therefore b^{-1} and $(cb^{-1}c')^{-1}$ are self adjoint. Similarly,

$$\begin{aligned}ax&=ab^{-1}(a'-c'(cb^{-1}c')^{-1}cb^{-1}a')\\&=x'bx.\end{aligned}$$

Therefore, $ax-\frac{1}{2}x'bx=\frac{1}{2}ax$, completing the proof. □

Lemma 2. *For firms competing $f\in\{p,b,c\}$, the regional production function is $Y_i^f(\ell,m)=z_im^\eta\ell^{1-\eta}\Phi^f(m)$, where $z_i\equiv\mathbb{E}[z_{isn}^{1/\eta}]^\eta$. Furthermore, $\Phi^f(m)\leq\Phi^p(m)$ for all f and m.*

Proof. This result follows immediately from Lemma 6 and noting that as perfect competition is efficient, it must produce the most conditional on the number of firms and workers. □

Lemma 3. *For firms competing $f \in \{b, c\}$, total wage compensation in location i can be written,*

$$w_i \ell_i = (1-\eta) z_i (m_i)^{\eta} (\ell_i)^{1-\eta} \left(\Phi^f(m_i) + \Psi^f(m_i) \right),$$

where $\Psi^f(m_i) \leq 0$.

Proof. This result follows from Propositions 4 and 5 and noting that wages must have a markdown. □

Lemma 4. *Expected sectoral HHI, weighted by average productivity shock, converges to 0 as the number of firms goes to infinity. More precisely,* $\psi_N \to 0$ *as* $N \to \infty$ *where* $\psi_N \equiv \mathbb{E}\left[\frac{\sum_{n \in \mathcal{N}} z_{isn}^{1/\eta}}{N z_i^{1/\eta}} \frac{\sum_{n \in \mathcal{N}} (z_{isn}^{1/\eta})^2}{\left(\sum_{n \in \mathcal{N}} z_{isn}^{1/\eta}\right)^2} \middle| N \right]$.

Proof. Note that because $1 - F_{iz}$ is regularly varying, there exists α and slowly varying function L[3] such that $1 - F_{iz}(x) = x^{-\alpha} L(x)$.

Then there are two cases: $\mathbb{E}[z_{isn}^{2/\eta}]$ exists and $\mathbb{E}[z_{isn}^{2/\eta}]$ does not exist. We will take each case in turn.

Suppose that $\mathbb{E}[z_{isn}^{2/\eta}]$ exists. Then we can write ψ_N,

$$\psi_N = \frac{1}{z_i^{1/\eta}} \mathbb{E}\left[\frac{1}{N} \frac{N}{\sum_{n \in \mathcal{N}} z_{isn}^{1/\eta}} \frac{\sum_{n \in \mathcal{N}} (z_{isn}^{1/\eta})^2}{N} \middle| N \right].$$

By the strong law of large numbers, $\frac{1}{N} \to 0$, $\frac{N}{\sum_{n \in \mathcal{N}} z_{isn}^{1/\eta}} \to \frac{1}{\mathbb{E}[z_{isn}^{1/\eta}]}$, and $\frac{\sum_{n \in \mathcal{N}} (z_{isn}^{1/\eta})^2}{N} \to \mathbb{E}[z_{isn}^{2/\eta}]$ almost surely. Therefore, the integrand converges to 0 almost surely, and $\psi_N \to 0$.

Suppose that $\mathbb{E}[z_{isn}^{2/\eta}]$ does not exist. Then we can rewrite ψ_N as

$$\psi_N = \mathbb{E}\left[\frac{a_N}{N^2} \frac{\sum_{n \in \mathcal{N}} (z_{isn}^{1/\eta})^2}{a_N} \frac{N}{\sum_n z_{isn}^{1/\eta}} \right],$$

where a_N is defined so that $\mathbb{P}(z_{isn}^{2/\eta} > a_N) = N^{-1}$. By Lévy's theorem,[4] $\frac{1}{a_N} \left(\sum_n (z_{isn}^{1/\eta})^2 \right)$ converges in distribution to a non-degenerate distribution, $\frac{N}{\sum_n z_{isn}^{1/\eta}} \to \frac{1}{\mathbb{E}[z_{isn}^{1/\eta}]}$ almost surely. That simply leaves a_N / N^2. Note that

$$\begin{aligned} \frac{a_N}{N^2} &= a_N \mathbb{P}\left(z_{isn}^{2/\eta} > a_N \right)^2 \\ &= a_N a_N^{-\alpha} L(a_N^{1/2})^2. \end{aligned}$$

[3] A function is slowly varying if for every $a > 0$, $\frac{L(ax)}{L(x)} \to 1$ as $x \to \infty$.

[4] See Durrett (2019).

This converges to 0 as $a_N \to \infty$ if $\alpha > 1$. But note that since the mean exists, α must be greater than 1. Further, note that $a_N \to \infty$ as $N \to \infty$; otherwise, the variance would exist. Thus, the integrand must converge to 0 in distribution and $\psi_N \to 0$. □

Lemma 5. *Average sectoral HHI has the following properties:*

(i) $\frac{\partial}{\partial \log m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds\right] < 0$*; and*

(ii) $\frac{\partial}{\partial \log m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds\right] \to 0$ *as* $m \to \infty$.

Proof. Recall that $\overline{\ell}_{isn} = \left(\frac{(1-\eta)z_{isn}}{\overline{w}_i}\right)^{\frac{1}{\eta}}$ where $\overline{w}_i = (1-\eta)\ell_i^{-\eta}m_i^{\eta}z_i$. Therefore,

$$\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 ds = \int_0^1 \frac{\ell_i m_i^{-1} z_i^{-\frac{1}{\eta}}\sum_{n\in\mathcal{N}_{is}} z_{isn}^{\frac{1}{\eta}}}{\ell_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{z_{isn}^{\frac{1}{\eta}}}{\sum_{n'\in\mathcal{N}_{is}} z_{isn'}^{\frac{1}{\eta}}}\right)^2 ds$$
$$= \int_0^1 \frac{N_{is}}{m_i}\frac{\sum_{n\in\mathcal{N}_{is}} z_{isn}^{1/\eta}}{N_{is} z_i^{1/\eta}}\frac{\sum_{n\in\mathcal{N}_{is}}(z_{isn}^{1/\eta})^2}{\left(\sum_{n\in\mathcal{N}_{is}} z_{isn}^{1/\eta}\right)^2} ds$$

We can then break this integral up into an expectation over the number of firms in the sector and, conditional number of firms, the expected productivity shocks. We will denote $\psi_N = \mathbb{E}\left[\frac{\sum_{n\in\mathcal{N}} z_{isn}^{1/\eta}}{N z_i^{1/\eta}}\frac{\sum_{n\in\mathcal{N}}(z_{isn}^{1/\eta})^2}{\left(\sum_{n\in\mathcal{N}} z_{isn}^{1/\eta}\right)^2}|N\right]$. Then we have

$$\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2 = \sum_{N=0}^{\infty}\frac{N}{m}\psi_N\frac{m^N e^{-m}}{N!}.$$

Then we can take the derivative with respect to m,

$$\frac{\partial}{\partial m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2\right] = \frac{\partial}{\partial m}\left[\sum_{N=1}^{\infty}\frac{N}{m}\psi_N\frac{m^N e^{-m}}{N!}\right]$$
$$= \sum_{N=1}(N-1)N\psi_N\frac{m^{N-2}e^{-m}}{N!} - \sum_{N=1}N\psi_N\frac{m^{N-1}e^{-m}}{N!}$$
$$= \sum_{N=2}\psi_N\frac{m^{N-2}e^{-m}}{(N-2)!} - \sum_{N=2}\psi_{N-1}\frac{m^{N-2}e^{-m}}{(N-2)!}$$
$$= \sum_{N=2}(\psi_N - \psi_{N-1})\frac{m^{N-2}e^{-m}}{(N-2)!} < 0,$$

where the inequality comes from the fact that ψ_N is decreasing so $\psi_N - \psi_{N-1} < 0$. Therefore, HHI is decreasing.

We next turn to proving Lemma 5.2, that $m\frac{\partial}{\partial m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2\right]$ converges to zero as $m \to \infty$.

We start by showing that $\frac{\partial}{\partial m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2\right] \to 0$. Take $\epsilon > 0$. Since $\psi_N \to 0$, there is some $\overline{N}$ such that for $N \geq \overline{N}$, $\psi_N < \frac{\epsilon}{2}$. Notice that $\frac{m^{x-2}e^{-m}}{(x-2)!} \in (0,1)$. And also notice that $\frac{m^{x-2}e^{-m}}{(x-2)!} \to 0$ as $m \to \infty$.

Thus, there is a $\overline{m}$ such that for $m > \overline{m}$, $\frac{m^{x-2}e^{-m}}{(x-2)!} < \frac{1}{\psi_1}\frac{\epsilon}{2}$ for all $x \leq \overline{N}$. Therefore, for $m > \overline{m}$,

$$\begin{aligned}
\frac{\partial}{\partial m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2\right] &= \sum_{N=2}^{\infty}\left(\psi_N - \psi_{N-1}\right)\frac{m^{x-2}e^{-m}}{(x-2)!} \\
&> \sum_{N=2}^{\overline{N}}(\psi_N - \psi_{N-1})\frac{m^{x-2}e^{-m}}{(x-2)!} + \sum_{N=\overline{N}+1}^{\infty}(\psi_N - \psi_{N-1}) \\
&= \sum_{N=2}^{\overline{N}}(\psi_N - \psi_{N-1})\frac{m^{x-2}e^{-m}}{(x-2)!} - \psi_{\overline{N}} \\
&> \sum_{N=2}^{\overline{N}}(\psi_N - \psi_{N-1})\frac{1}{\psi_1}\frac{\epsilon}{2} - \frac{\epsilon}{2} \\
&= -\frac{\epsilon}{2} - \frac{\epsilon}{2}.
\end{aligned}$$

Furthermore, the second derivative is positive for sufficiently large m,

$$\frac{\partial^2}{\partial m^2}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2\right] = \sum_{N=3}^{\infty}\left((\psi_N - \psi_{N-1}) - (\psi_{N-1} - \psi_{N-2})\right)\frac{m^{N-3}e^{-m}}{(N-3)!} > 0,$$

because ψ_N is convex for sufficiently large N. Therefore, we have a function $f(x)$ such that $f(x) \to 0$, $f'(x) \to 0$ and $f''(x) > 0$. It then follows that $xf'(x) \to 0$. That is $m\frac{\partial}{\partial m}\left[\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\left(\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\right)^2\right] \to 0$.

To see this, suppose that $xf'(x)$ does not converge to zero. Then there is a $\epsilon > 0$ and a sequence of $x_n \to \infty$ such that $x_n f'(x_n) < -\epsilon$. As $f''(x) > 0$, it follows that $f'(x) < f'(x_n) < -\frac{\epsilon}{x_n}$ for all $x < x_n$. Therefore,

$$
\begin{aligned}
f(x) &= f(0) + \int_0^x f'(t)dt \\
&< f(0) + \sum_{n=1}^{N:x_N<x} \int_{x_{n-1}}^{x_n} f'(x_n)dt + \int_{X_N}^{x} f'(x)dt \\
&< f(0) - \sum_{n=1}^{N:x_N<x} (x_n - x_{n-1}) \frac{\epsilon}{x_n} + (x - x_N) f'(x) \\
&\to f(0) - \sum_{n=1}^{\infty} (x_n - x_{n-1}) \frac{\epsilon}{x_n}
\end{aligned}
$$

But then

$$
\sum_{n=1}^{\infty} (x_n - x_{n-1}) \frac{\epsilon}{x_n} \approx \epsilon \int_0^{\infty} \frac{1}{x} dx \to \infty.
$$

Therefore, this contradicts, $f(x) \to 0$ so $xf'(x)$ must converge to 0. □

Lemma 6. *The regional production function is $Y_i(\ell, m) = z_i m^{\eta} \ell^{1-\eta} \tilde{\Phi}(m)$, where $z_i \equiv \mathbb{E}[z_{isn}^{1/\eta}]^{\eta}$ and $\tilde{\Phi}(m)$ is given by,*

$$
\begin{aligned}
\tilde{\Phi}(m) \equiv \mathbb{E}[a_{sn}(\omega)] &+ (1-\eta) \int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n \in \mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}[\hat{a}_{sn}(\omega) \hat{\ell}_{sn}(\omega)] ds \\
&- \eta \frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n \in \mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}[\hat{\ell}_{sn}(\omega)^2] ds \\
&- \frac{1}{\kappa} \frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_s}{\ell} \sum_{n \in \mathcal{N}_s} \frac{\overline{\ell}_{sn}}{\overline{\ell}_s} \mathbb{E}\left[\left(\hat{\ell}_{sn}(\omega) - \hat{\ell}_{sn} \right)^2 \right] ds \\
&- \left(\frac{1}{\nu} - \frac{1}{\kappa} \right) \frac{1-\eta}{2} \int_0^1 \frac{\overline{\ell}_s}{\ell} \mathbb{E}\left[\left(\hat{\ell}_s(\omega) - \hat{\ell}_s \right)^2 \right] ds. \qquad (14)
\end{aligned}
$$

Proof. Expected production is $\int_0^1 \sum_{n \in \mathcal{N}_{is}} z_{isn} a_{isn}(\omega) \ell_{isn}(\omega)^{1-\eta} ds$. We will do a second order approximation around the point $\log \tilde{a}_{isn}(\omega) = \log \tilde{A}_{is}(\omega) = 0$.

We start by characterizing the solution at that point. We find that there is some $\overline{w}_i$ such that

$$
\overline{w}_i = (1-\eta) z_{isn} (\overline{\ell}_{isn})^{-\eta},
$$

so that $\overline{\ell}_{isn} = \left(\frac{(1-\eta)z_{isn}}{\overline{w}_i}\right)^{\frac{1}{\eta}}$. Then labor clearing requires

$$\ell_i = \int_0^1 \ell_i \overline{L}_{is} ds = \int_0^1 \sum_{n\in\mathcal{N}_{is}} \overline{\ell}_{isn} ds = \int_0^1 \sum_{n\in\mathcal{N}_{is}} \left(\frac{(1-\eta)z_{isn}}{\overline{w}_i}\right)^{\frac{1}{\eta}} ds.$$

This implies that $\overline{w}_i = (1-\eta)\ell_i^{-\eta} m_i^{\eta} \mathbb{E}\left[z_{isn}^{1/\eta}\right]^{\eta}$. Then production is

$$\overline{Y}_i = \int_0^1 \sum_{n\in\mathcal{N}_{is}} z_{isn} \left(\frac{(1-\eta)z_{isn}}{\overline{w}_i}\right)^{\frac{1-\eta}{\eta}} ds = z_i \ell_i^{1-\eta} m_i^{\eta},$$

where $z_i = \mathbb{E}\left[z_{isn}^{1/\eta}\right]^{\eta}$.

Taking the log second order approximation to production gives

$$Y_i \approx z_i (m_i)^{\eta} (\ell_i)^{1-\eta} \int_0^1 \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_i} \Bigg(1 + \left(\hat{a}_{isn}(\omega) + (1-\eta)\hat{\ell}_{isn}(\omega)\right) \\ + \frac{1}{2}\left(\hat{a}_{isn}(\omega) + (1-\eta)\hat{\ell}_{isn}(\omega)\right)^2\Bigg) ds.$$

To transform this to be completely second order, we do a second order approximation to the labor constraints,

$$-\frac{1}{\kappa}\hat{\ell}_{is} + \frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega) + \frac{1}{2}\left(-\frac{1}{\kappa}\hat{\ell}_{is} + \frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega)\right)^2 = \\ \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \left[-\frac{1}{\kappa}\hat{\ell}_{isn} + \frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega) + \frac{1}{2}\left(-\frac{1}{\kappa}\hat{\ell}_{isn} + \frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega)\right)^2\right],$$

$$0 = \int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\left[-\frac{1}{\nu}\hat{\ell}_{is} + \frac{1+\nu}{\nu}\hat{\ell}_{is}(\omega) + \frac{1}{2}\left(-\frac{1}{\nu}\hat{\ell}_{is} + \frac{1+\nu}{\nu}\hat{\ell}_{is}(\omega)\right)^2\right] ds,$$

$$\hat{\ell}_{is} + \frac{1}{2}\hat{\ell}_{is}^2 = \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\left(\hat{\ell}_{isn} + \frac{1}{2}\hat{\ell}_{isn}^2\right),$$

and

$$0 = \int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\left(\hat{\ell}_{is} + \frac{1}{2}\hat{\ell}_{is}^2\right) ds,$$

where we use the fact $\hat{\ell}_{isn}(\omega) = \hat{L}_{isn}(\omega)$.

Then we can transform $\int_0^1 \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_i}\hat{\ell}_{isn}(\omega)ds$ to second order. That is,

$$
\begin{aligned}
\int_0^1 \sum_{n\in\mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_i}\hat{\ell}_{isn}(\omega)ds =& -\frac{1}{2}\frac{\kappa}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\left(-\frac{1}{\kappa}\hat{\ell}_{isn}+\frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega)\right)^2 ds \\
&+\frac{1}{2}\frac{\kappa}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\left(-\frac{1}{\kappa}\hat{\ell}_{is}+\frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega)\right)^2 ds \\
&+\frac{1}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{\ell}_{isn}ds-\frac{1}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\hat{\ell}_{is}ds \\
&+\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\hat{\ell}_{is}(\omega)ds \\
=& -\frac{1}{2}\frac{\kappa}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\left(-\frac{1}{\kappa}\hat{\ell}_{isn}+\frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega)\right)^2 ds \\
&+\frac{1}{2}\frac{\kappa}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\left(-\frac{1}{\kappa}\hat{\ell}_{is}+\frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega)\right)^2 ds \\
&-\frac{1}{2}\frac{\nu}{1+\nu}\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\left(-\frac{1}{\nu}\hat{\ell}_{is}+\frac{1+\nu}{\nu}\hat{\ell}_{is}(\omega)\right)^2 ds \\
&+\frac{1}{1+\nu}\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\hat{\ell}_{is}ds \\
&+\frac{1}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{\ell}_{isn}ds-\frac{1}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\hat{\ell}_{is}ds \\
=& -\frac{1}{2}\frac{\kappa}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\left(-\frac{1}{\kappa}\hat{\ell}_{isn}+\frac{1+\kappa}{\kappa}\hat{\ell}_{isn}(\omega)\right)^2 ds \\
&+\frac{1}{2}\frac{\kappa}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\left(-\frac{1}{\kappa}\hat{\ell}_{is}+\frac{1+\kappa}{\kappa}\hat{\ell}_{is}(\omega)\right)^2 ds \\
&-\frac{1}{2}\frac{\nu}{1+\nu}\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\left(-\frac{1}{\nu}\hat{\ell}_{is}+\frac{1+\nu}{\nu}\hat{\ell}_{is}(\omega)\right)^2 ds \\
&-\frac{1}{2}\frac{1}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{\ell}_{isn}^2 ds \\
&+\frac{1}{2}\frac{1}{1+\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\ell_i}\hat{\ell}_{is}^2 ds \\
&-\frac{1}{2}\frac{1}{1+\nu}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\hat{\ell}_{is}^2 ds \\
=& -\frac{1}{2}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i}\sum_{n\in\mathcal{N}_{is}}\frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}}\hat{\ell}_{isn}(\omega)^2 ds
\end{aligned}
$$

$$
- \frac{1}{2}\frac{1}{\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn}\right)^2 ds
- \frac{1}{2}\left(\frac{1}{\nu} - \frac{1}{\kappa}\right)\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_{is}} \left(\hat{\ell}_{is}(\omega) - \hat{\ell}_{is}\right)^2 ds.
$$

Substituting this into the expression for the log second order approximation to production gives

$$
\begin{aligned}
\frac{Y_i}{z_i(m_i)^\eta(\ell_i)^{1-\eta}} \approx\ & 1 + \mathbb{E}[\hat{a}_{isn}(\omega)] + \frac{1}{2}\mathbb{E}[\hat{a}_{isn}(\omega)^2] \\
& + (1-\eta)\int_0^1 \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\ell_i} \hat{a}_{isn}(\omega)\hat{\ell}_{isn}(\omega) ds \\
& + \frac{(1-\eta)^2}{2}\int_0^1 \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\ell_i} \hat{\ell}_{isn}(\omega)^2 ds \\
& - \frac{1-\eta}{2}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \hat{\ell}_{isn}(\omega)^2 ds \\
& - \frac{1-\eta}{2}\frac{1}{\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn}\right)^2 ds \\
& - \frac{1-\eta}{2}\left(\frac{1}{\nu} - \frac{1}{\kappa}\right)\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_{is}} \left(\hat{\ell}_{is}(\omega) - \hat{\ell}_{is}\right)^2 ds \\
\approx\ & \mathbb{E}[a_{isn}(\omega)] + (1-\eta)\int_0^1 \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\ell_i} \hat{a}_{isn}(\omega)\hat{\ell}_{isn}(\omega) ds \\
& - \frac{1-\eta}{2}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \hat{\ell}_{isn}(\omega)^2 ds \\
& - \frac{1-\eta}{2}\eta\frac{1}{\kappa}\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_i} \sum_{n \in \mathcal{N}_{is}} \frac{\overline{\ell}_{isn}}{\overline{\ell}_{is}} \left(\hat{\ell}_{isn}(\omega) - \hat{\ell}_{isn}\right)^2 ds \\
& - \frac{1-\eta}{2}\left(\frac{1}{\nu} - \frac{1}{\kappa}\right)\int_0^1 \frac{\overline{\ell}_{is}}{\overline{\ell}_{is}} \left(\hat{\ell}_{is}(\omega) - \hat{\ell}_{is}\right)^2 ds,
\end{aligned}
$$

where we use the fact that to log second order, $\mathbb{E}[a_{sn}(\omega)] \approx 1 + \mathbb{E}[\hat{a}_{sn}(\omega)] + \frac{1}{2}\mathbb{E}[\hat{a}_{sn}(\omega)^2]$.

□

3 Equilibrium Conditions

In this section, we restate the equilibrium conditions.

3.1 Competitive Equilibrium

The competitive equilibrium is characterized by the following equations.

Migration Decision.

$$\ell_i = \left(\frac{\overline{u}_i w_i}{u}\right)^{\theta} \ell; \qquad u = \left(\sum_{i \in \mathcal{I}} (\overline{u}_i w_i)^{\theta}\right)^{\frac{1}{\theta}}$$

Competitive Wages.

$$w_i = (1-\eta) z_i (m_i)^{\eta} (\ell_i)^{-\eta} \Phi^c(m_i)$$

Free Entry.

$$\psi_i = \eta z_i m_i^{\eta-1} (\ell_i)^{1-\eta} \Phi^c(m_i)$$

3.2 Imperfect Competition

With imperfect competition $f \in \{b, c\}$, workers are no longer paid their competitive wages.

Adjusted Wages.

$$w_i = (1-\eta) z_i (m_i)^{\eta} (\ell_i)^{-\eta} (\Phi^f(m_i) + \Psi^f(m_i))$$

Adjusted Free Entry.

$$\begin{aligned} \psi_i &= \frac{z_i (m_i)^{\eta} (\ell_i)^{1-\eta} \Phi^f(m_i) - (1-\eta) z_i (m_i)^{\eta} (\ell_i)^{1-\eta} (\Phi^f(m_i) + \Psi^f(m_i))}{m_i} \\ &= \eta (m_i)^{\eta-1} (\ell_i)^{1-\eta} \Phi^f(m_i) \left(1 - \frac{1-\eta}{\eta} \frac{\Psi^f(m_i)}{\Phi^f(m_i)}\right) \end{aligned}$$

3.3 Planner's Solution

We assume the planner pays for the entry and wages subsidies with proportional taxes on all workers. Therefore, the migration decision does not change. However, for all

$f \in \{p, b, c\}$ the free entry condition is replaced with the first-best level of entry condition,

$$\psi_i = \eta z_i m_i^{\eta-1} (\ell_i)^{1-\eta} \Phi^f(m_i) \left(1 + \frac{1}{\eta} \frac{\partial \log \Phi^f(m_i)}{\partial \log m}\right).$$

Furthermore, wages must be such that workers get their marginal product,

$$w_i = (1-\eta) z_i (m_i)^{\eta} (\ell_i)^{-\eta} \Phi^f(m_i)$$

References

Benigno, P. and Woodford, M. (2003). Optimal monetary and fiscal policy: A linear-quadratic approach. *NBER macroeconomics annual*, 18:271–333.

Benigno, P. and Woodford, M. (2012). Linear-quadratic approximation of optimal policy problems. *Journal of Economic Theory*, 147(1):1–42.

Durrett, R. (2019). *Probability: theory and examples*, volume 49. Cambridge university press.